STAT 515 hw 8

Hypothesis testing for the mean and proportion by comparing test statistics to critical values

1. Each of fifteen students measured a cup of flour by fluffing the flour and then scooping it into a measuring cup with a spoon until it was slightly mounded above the brim, and then scraping away the excess. The student then measured the weight of the cup of flour in grams, resulting in the following measurements:

   `128 151 125 127 128 117 130 129 127 154 162 130 154 131 140.`

   Assume that the measurements follow a Normal distribution.

   (a) Write down the hypotheses of interest for the following chefs:

   i. Chef Boyardee wishes to know whether the mean weight of cups of flour measured in this way exceeds 120 grams.

   > Letting μ represent the mean weight of cups of flour measured in this way, Chef Boyardee is interested in testing
   >
   > $$H_0\colon\ \mu \leq 120 \text{ versus } H_1\colon\ \mu > 120.$$

   ii. Chef Bertolli wishes to know whether the mean weight of cups of flour measured in this way is less than 120 grams.

   > Chef Bertolli is interested in testing
   >
   > $$H_0\colon\ \mu \geq 120 \text{ versus } H_1\colon\ \mu < 120.$$

   iii. Chef Ragu wishes to know whether the mean weight of cups of flour measured in this way differs from 120 grams.

   > Chef Ragu is interested in testing
   >
   > $$H_0\colon\ \mu = 120 \text{ versus } H_1\colon\ \mu \neq 120.$$

   (b) Suppose the true mean weight of cups of flour measured in this way is 115 grams, and that each of the chefs in part (a) conducts a study to answer his question.

   i. For which of the chefs in part (a) is a Type I error possible?

   > The only chef for whom the null hypothesis is true is Chef Boyardee, so only he can make a Type I error.

   ii. For which of the chefs in part (a) is a Type II error possible?

For Chef Bertolli and Chef Ragu the null hypothesis is false, so they are liable to making a Type II error. This would occur if the data did not contain sufficient evidence against the null to reject it.

(c) Assume that the standard deviation of the weights of cups of flour measured in this way is $\sigma = 12$.

i. Give a 95% confidence interval for the mean weight of cups of flour measured in this way.

A 95% confidence interval for the mean weight of cups of flour measured in this way is

$$135.53 \pm 1.96 \cdot \frac{13.31}{\sqrt{15}} = (128.8, 142.3).$$

ii. Give the value of the test statistic for testing the hypotheses of the chefs in part (a).

The value of the test statistic is

$$Z_{\text{test}} = \frac{\bar{X}_n - 120}{\sigma/\sqrt{n}} = \frac{135.53 - 120}{12/\sqrt{15}} = 5.01.$$

iii. State, for each chef in part (a), whether he rejects his null hypotheses at the $\alpha = 0.05$ significance level based on the class data. Explain your answer.

We have

1. For Chef Boyardee, the $\alpha = 0.05$ critical value is $z_{0.05} = 1.645$. Since the value of the test statistic $Z_{\text{test}} = 5.01$ exceeds the critical value, we reject H_0 and conclude that the mean weight of cups of flour measured in this way exceeds 120 grams.

2. For Chef Bertolli, the $\alpha = 0.05$ critical value is $-z_{0.05} = -1.645$. Since the value of the test statistic $Z_{\text{test}} = 5.01$ does not fall below the critical value, we fail to reject H_0.

3. For Chef Ragu, the $\alpha = 0.05$ critical value is $z_{0.025} = 1.96$. Since the absolute value of the test statistic $Z_{\text{test}} = 5.01$ exceeds the critical value, we reject H_0 and conclude that the mean weight of cups of flour measured in this way is not equal to 120 grams; moreover, it exceeds 120 grams, since the value of the test statistic is positive.

(d) Assume that the standard deviation of the weights of cups of flour measured in this way is unknown and use the sample standard deviation.

i. Give a 95% confidence interval for the mean weight of cups of flour measured in this way.

The interval is given by

$$135.53 \pm 2.144787 \cdot \frac{13.31}{\sqrt{15}} = (128.2, 142.9),$$

where $2.144787 = t_{15-1,0.025}$.

ii. Give the value of the test statistic for testing the hypotheses of the chefs in part (a).

The test statistic for testing the hypotheses of interest to the chefs is

$$T_{\text{test}} = \frac{\bar{X}_n - 120}{S_n/\sqrt{n}} = \frac{135.53 - 120}{13.31/\sqrt{15}} = 4.52.$$

iii. State, for each chef in part (a), whether he rejects his null hypotheses at the $\alpha = 0.05$ significance level based on the class data. Explain your answer.

1. For Chef Boyardee, the $\alpha = 0.05$ critical value is $t_{15-1,0.05} = 1.76131$. Since the value of the test statistic $T_{\text{test}} = 4.52$ exceeds the critical value, we reject H_0 and conclude that the mean weight of cups of flour measured in this way exceeds 120 grams.
2. For Chef Bertolli, the $\alpha = 0.05$ critical value is $-t_{15-1,0.05} = -1.76131$. Since the value of the test statistic $T_{\text{test}} = 4.52$ does not fall below the critical value, we fail to reject H_0.
3. For Chef Ragu, the $\alpha = 0.05$ critical value is $t_{15-1,0.025} = 2.144787$. Since the absolute value of the test statistic $T_{\text{test}} = 4.52$ exceeds the critical value, we reject H_0 and conclude that the mean weight of cups of flour measured in this way is not equal to 120 grams; moreover, it exceeds 120 grams, since the value of the test statistic is positive.

2. Rosenzweig et al. (1972) subjected pairs of rats to live in "impoverished" versus "enriched" surroundings, where each pair of rats was taken from a single litter to ensure genetic similarity. In the enriched environment, rats lived among other rats in large cages and were furnished with new playthings everyday, whereas in the impoverished environment, each rat lived alone in an unfurnished cage. At the end of an experimental period "the rats were sacrificed and their brains removed" and their cortexes weighed, resulting in the following weights in milligrams (partial data are given as presented on pg. 453 of Freedman et al. (1991)):

Pair	Enriched	Impoverished	Pair	Enriched	Impoverished
1	689	657	7	664	600
2	656	623	8	647	640
3	668	652	9	694	605
4	660	654	10	633	635
5	679	658	11	653	642
6	663	646			

(a) Write down the hypotheses in which the following scientists are interested:

i. Rosenzweig wants to know if the enriched environment leads to heavier cortexes.

> Letting μ represent the mean difference in the cortex weight of rats in enriched versus impoverished environments, Rosenzweig is interested in testing
>
> $$H_0\colon\ \mu \leq 0 \text{ versus } H_1\colon\ \mu > 0.$$

ii. Bennett wants to know if the enriched environment results in lighter cortexes.

> Bennett is interested in testing
>
> $$H_0\colon\ \mu \geq 0 \text{ versus } H_1\colon\ \mu < 0.$$

iii. Diamond wants to know if being in an enriched versus an impoverished environment has any effect on the weights of rat cortexes.

> Diamond is interested in testing
>
> $$H_0\colon\ \mu = 0 \text{ versus } H_1\colon\ \mu \neq 0.$$

(b) Suppose the mean difference (enriched minus impoverished) is 5 milligrams.

i. Which scientists could make a Type I error?

> The only scientist for whom the null hypothesis is true under $\mu = 5$ is Bennett, so Bennett is the only one at risk of making a Type I error.

ii. Which scientists could make a Type II error?

> The alternate hypotheses of both Rosenzweig and Diamond are true under $\mu = 5$, so each of these scientists is liable to making a Type II error. This would happen if the evidence in the data were not strong enough to justify the rejection of H_0.

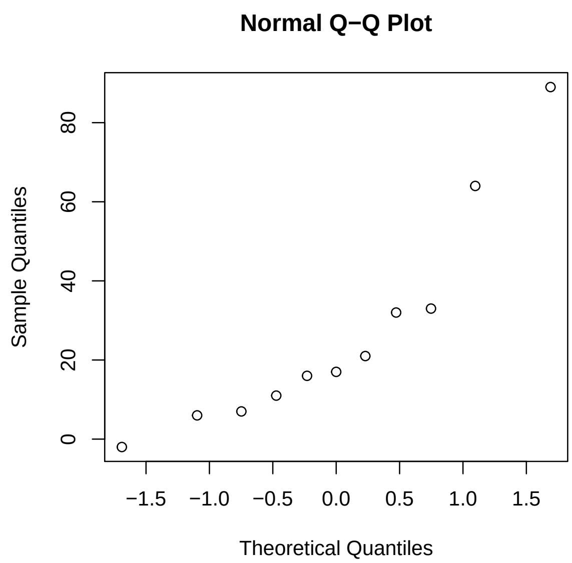
(c) Calculate the sample mean and standard deviation of the differences between the cortex weights of the rats in the enriched versus impoverished environments (enriched minus impoverished).

We have

$$\bar{X}_n = 26.73 \quad \text{and} \quad S_n = 27.328.$$

(d) Make a QQ plot of the differences between the cortex weights of the rats in the enriched versus impoverished environments (enriched minus impoverished). Comment on whether the differences appear to be Normally distributed.

The Normal Q-Q plot looks like this:



There is some doubt as to whether the data come from a Normal distribution, since the points appear to follow a curve rather than a straight line.

(e) Assuming that the differences in cortex weight follow a Normal distribution, give the value of the test statistic for testing the hypotheses of the scientist in part (a).

Since $\mu_0 = 0$, the test statistic is

$$T_{\text{test}} = \frac{\bar{X}_n - 0}{S_n/\sqrt{n}} = \frac{26.73}{27.328/\sqrt{11}} = 3.24.$$

(f) Assuming that the differences in cortex weight follow a Normal distribution, state, for each scientist in part (a), whether he or she rejects the null hypothesis at the $\alpha = 0.05$ level.

We have:

1. For Rosenzweig, the critical value is $t_{10,0.05} =$ `qt(.95,10)` $= 1.812461$. Since $T_{\text{test}} =$

3.24 is greater than 1.812461, we reject H_0 and conclude that the enriched environment lead to increased cortex weight in the rats.

2. For Bennett, we would fail to reject H_0, since our test statistic does not fall below the critical value $-t_{10,0.05} = -1.812461$.

3. For Diamond, the critical value is $t_{10,0.025}$ = `qt(0.975,10)` = 2.228139. Since $T_{\text{test}} = 3.24$ is greater than 2.228139, we reject H_0 and conclude that the enriched environment makes a difference in the cortex weight; moreover, we can say that the difference is positive, so that the enriched environment leads to increased cortex weight in the rats.

(g) Assuming that the differences in cortex weight follow a Normal distribution, build a 95% confidence interval for the mean difference in cortex weight between rats in an enriched versus an impoverished environment.

A 95% confidence interval for the mean difference in cortex weight between rats in an enriched versus an impoverished environment is given by

$$26.73 \pm \underbrace{t_{10,0.025}}_{2.2281} \frac{27.328}{\sqrt{11}} = (8.37, 45.09).$$

3. Sea lampreys are an invasive species in the Great Lakes—primarily in Lake Michigan—whose threat to the population of fish there began causing alarm in the 1930s. Sea lampreys attach themselves to fish and feed on them. In 1947, Vernon C. Applegate caught 3,700 white or redhorse suckers (kinds of fish) in a weir in a tributary to Lake Michigan, among which 259 had scars inflicted by sea lampreys (Applegate, 1950).

   (a) Suppose the Department of Conservation will take measures to control the sea lamprey population in a tributary if the proportion of scarred white or redhorse suckers in a tributary is determined to exceed 0.06.

   i. State the null and alternate hypotheses of interest to the Department of Conservation.

   Letting p represent the true propotion of white or redhorse suckers in the tributary on which sea lampreys have inflicted scars, the Department of Conservation is interested in the hypotheses

   $$H_0\colon\ p \leq 0.06 \text{ versus } H_1\colon\ p > 0.06.$$

   ii. Given the data of Vernon C. Applegate, would the Department of Conservation take measures to control the sea lamprey population if using the significance level $\alpha = 0.05$?

   Setting $\hat{p}_n = 259/3700 = 0.07$, the test statistic for testing the hypotheses of interest to the Department of Conservation is

$$Z_{\text{test}} = \frac{0.07 - 0.06}{\sqrt{0.06(1-0.06)/3700}} = 2.561.$$

This number exceeds the $\alpha = 0.05$ critical value $z_{0.05} = 1.645$, so the Department of Conservation would choose to take measures to control the sea lamprey population.

iii. If using the significance level $\alpha = 0.005$?

The $\alpha = 0.005$ critical value is $z_{0.005} =$ `qnorm(0.995)` $= 2.576$, so the Department of Conservation would *not* choose to take measures to control the sea lamprey population.

(b) Suppose measures to control the sea lamprey population are already underway in a tributary and that the Department of Conservation will cease these measures if the proportion of scarred white or redhorse suckers in the tributary is determined to be less than 0.02.

i. State the null and alternate hypotheses of interest to the Department of Conservation.

The Department of Conservation is interested in the hypotheses

$$H_0\colon\ p \geq 0.02 \text{ versus } H_1\colon\ p < 0.02.$$

ii. Given the data of Vernon C. Applegate, would the Department of Conservation cease its measures of controlling the sea lamprey population if using the significance level 0.05?

Since the data lends support to the null hypothesis, $\hat{p}_n = 0.07$, there is no evidence in favor of H_1. Therefore we do not reject H_0. We can determine this without doing any calculations. If we were to do the calculations, however, we would obtain the test statistic

$$Z_{\text{test}} = \frac{0.07 - 0.02}{\sqrt{0.02(1-0.02)/3700}} = 21.72415.$$

Since this exceeds the $\alpha = 0.05$ critical value $-z_{0.05} = -1.645$, the null hypothesis is not rejected.

iii. If using the significance level $\alpha = 0.005$?

The decision is still the same. The Department of Conservation will not cease its measures.

(c) Given the data of Vernon C. Applegate, construct a 95% confidence interval for the proportion of white or redhorse suckers in the tributary on which sea lampreys have fed.

The 95% Wald-type confidence interval for the true proportion of white or redhorse suckers in the tributary on which sea lampreys have fed is given by

$$0.07 \pm 1.96\sqrt{\frac{0.07(1-0.07)}{3700}} = (0.062, 0.078).$$

4. Last summer you earned \$120 every Saturday at your job and you are planning to take the same job again this summer. Your friend has a job with wages coming mostly from tips, and brags about making \$222 on one Saturday. She can get you a job where she works this summer if you want. In order to base your decision on rigorous statistical inference, you collect from her the wages she earned on several Saturdays last summer, getting the amounts

115.15	117.34	222.09	204.05	80.04	78.46
162.54	172.99	181.03	135.68	192.70	122.28.

Let μ denote the expected earnings on a Saturday at the job your friend can get for you.

(a) You decide that if you can expect to earn over \$20 more on a Saturday working where your friend works, then you will ask her to get you a job there. What are the relevant hypotheses?

You are interested in the hypotheses

$$H_0\colon\ \mu \le 140 \text{ versus } H_1\colon\ \mu > 140.$$

(b) Suppose you really like your old job and you only want to switch if you are quite quite sure that you will make over \$20 more on average. Then, referring to the hypotheses in part (a), you should perform your test

A. with a large α to ensure a small probability of Type I error.

B. with a small α to ensure a small probability of Type I error.

C. with a large α to ensure a small probability of Type II error.

D. with a small α to to ensure a small probability of Type II error.

(c) Assuming your friend's wages are Normally distributed, perform the test of your hypotheses in part (a) at the $\alpha = 0.05$ significance level.

You have $\bar{X}_n = 148.6958$ and $S_n = 47.5174$ with $n = 12$, so that your test statistic is

$$Z_{\text{test}} = \frac{148.6958 - 140}{47.5174/\sqrt{12}} = 0.6339415.$$

This value does not exceed the $\alpha = 0.05$ significance level $t_{11,0.05} = 1.795885$, so you do not reject your null hypothesis. You conclude that it is not worth asking your friend for a job where she works.

(d) You discover that your old supervisor quit and you don't really want to work under the new supervisor. You decide that you will ask your friend to get you a job where she works unless you can conclude that you will make more on Saturdays, on average, at your current job. What are the relevant hypotheses now (still letting μ represent the amount of money you could expect to make at your friends job on a Saturday)?

> Still letting μ represent the amount of money you could expect to make at your friends job on a Saturday, you are now interested in the hypotheses
>
> $$H_0\colon\ \mu \geq 120 \text{ versus } H_1\colon\ \mu < 120.$$

(e) Interpret what it means if you commit a Type I error when testing the hypotheses in part (d).

> If you commit a Type I error, it means that your friend does not make any less than you, but you conclude that she does, so you keep your own job.

Optional (do not turn in) problems for additional study from McClave, J.T. and Sincich T. (2017) *Statistics*, 13th Edition: 8.33, 8.35, 8.38, 8.40, 8.60, 8.61, 8.62, 8.74, 8.76, 8.78, 8.80, 8.88.

References

Applegate, V. C. (1950). *Natural history of the sea lamprey, Petromyzon marinus, in Michigan.* PhD thesis, University of Michigan.

Freedman, D., Pisani, R. L., Purves, R., and Adhikari, A. (1991). *Statistics, Second Edition.* W.W. Norton & Company, Inc.

Rosenzweig, M. R., Bennett, E. L., and Diamond, M. C. (1972). Brain changes in response to experience. *Scientific American*, 226(2):22–29.