STAT 515 Lec 01 slides

Basics of sets, probability

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These slides are an instructional aid; their sole purpose is to display, during the lecture, definitions, plots, results, etc. which take too much time to write by hand on the blackboard. They are not intended to explain or expound on any material.

Statistics

Analyzing Data / Learning from Data

Probability

Sets / set theory

1. Basics of sets

2. Basics of probability

Experiment

An *experiment* is a process which generates an outcome such that there is

- (i) more than one possible outcome
- (ii) the set of possible outcomes is known
- (iii) the outcome is not known in advance

Sample space and sample points

- The *sample space* S of an experiment is the set of possible outcomes.
- The outcomes in a sample space S are called *sample points*.

Examples of experiments

① Roll a 6-sided die

$$S = \{1, 2, 3, 4, 5, 6\}$$

② Skipping a stone, counting # skips

$$S = \{0, 1, 2, \ldots\}$$

③ Spinning a top, timing it till it falls.

$$S = [0, \infty)$$

④ Proportion of freshmen in a class of size N.

$$S = \underbrace{\left\{\frac{0}{N}, \frac{1}{N}, \frac{2}{N}, \ldots, \frac{N}{N}\right\}}$$

Maybe treat this like $[0, 1]$.

(5) Today's max temp minus Yesterday's max temp.

$$S = (-\infty, \infty)$$

(6) Flip a coin

$$S = \{ \text{heads}, \text{tails} \}$$

(7) Blood type of randomly selected donor.

$$S = \{ A+, A-, O+, O- \ldots \}$$

Exercise: Give the sample space S for the following experiments:

1. Roll of a 6-sided die.
2. Skipping a stone and counting the skips.
3. Spinning a top and timing how long it spins.
4. Proportion of freshman in 515.
5. Today's max temp minus yesterday's max temp.
6. Blood type of a randomly selected student.

Event

An *event* is a collection of possible outcomes of an experiment, that is any subset of S (including S itself).

- Usually represent events with capital letters $A, B, C, \ldots$
- Say an event A occurs if the outcome is in the set A.
- So events are equivalent to sets. Can refer to events as sets, to sets as events.
- Often refer to members of sets as *elements* of the set.

Exercise: Express the following events as subsets of the sample space.

1. Roll of a 6-sided die: $S = \{1, 2, 3, 4, 5, 6\}$
   $A =$ *odd number rolled.* $A = \{1, 3, 5\}$
2. Skipping a stone and counting the skips.
   $A =$ *no skips.* $A = \{0\}$
3. Spinning a top and timing how long it spins: $S = [0, \infty)$
   $A =$ *less than one second.* $A = [0, 1)$
4. Proportion ~~...~~ of freshman in 515.
   $A =$ ~~...~~ over a third. $A = (\frac{1}{3}, 1]$
5. Today's max temp minus yesterday's max temp.
   $A =$ *at least 5 degrees warmer today* $A = [5, \infty)$
6. Blood type of a randomly selected student:
   $A =$ *has A antigen in RBCs.*

   $A = \{A+, A-, AB+, AB-\}$

Set containment and equality of sets

Let A and B be two events in a sample space.

1. We say A *is contained in* B if every point in A is in B. We write $\boxed{A \subset B.}$
2. We say *A is equal to B* if $A \subset B$ and $B \subset A$. We write $A = B$.

We may also express $A \subset B$ by saying *A is a subset of B*.

Example: Skip a stone and count the skips and let $\boxed{A = \{0\}}$ and $\boxed{B = \{0, 1, 2\}.}$
Then $A \subset B$, but $B \not\subset A$. So $A \neq B$.

$A \subset B$

$B \not\subset A$

sample space

Elementary set operations

- The *complement* A^c of a set A is the set of points in S which are not in A.
- The *union* $A \cup B$ of A and B is the set of points in either A or B or in both.
- The *intersection* $A \cap B$ of A and B is the set of points in both A and B.

Draw pictures.

Roll a 6-sided die:

$S = \{1, 2, 3, 4, 5, 6\}$

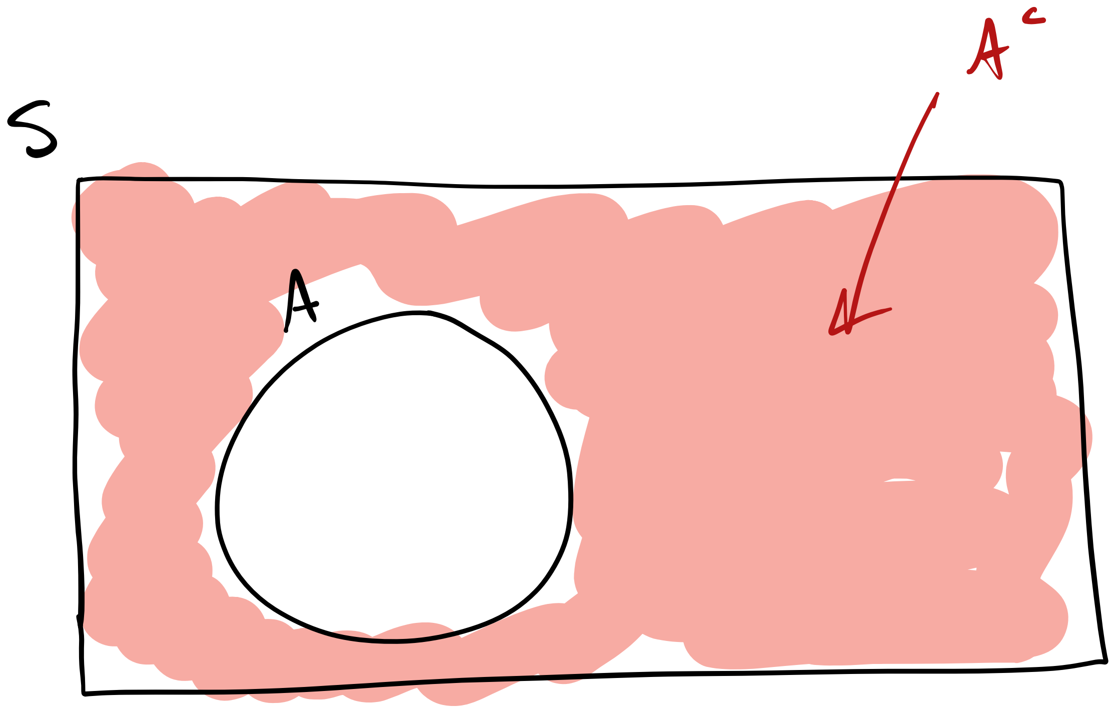
$A = \{1, 3, 5\} \xrightarrow{\text{comp}} A^c = \{2, 4, 6\} = S \setminus A$

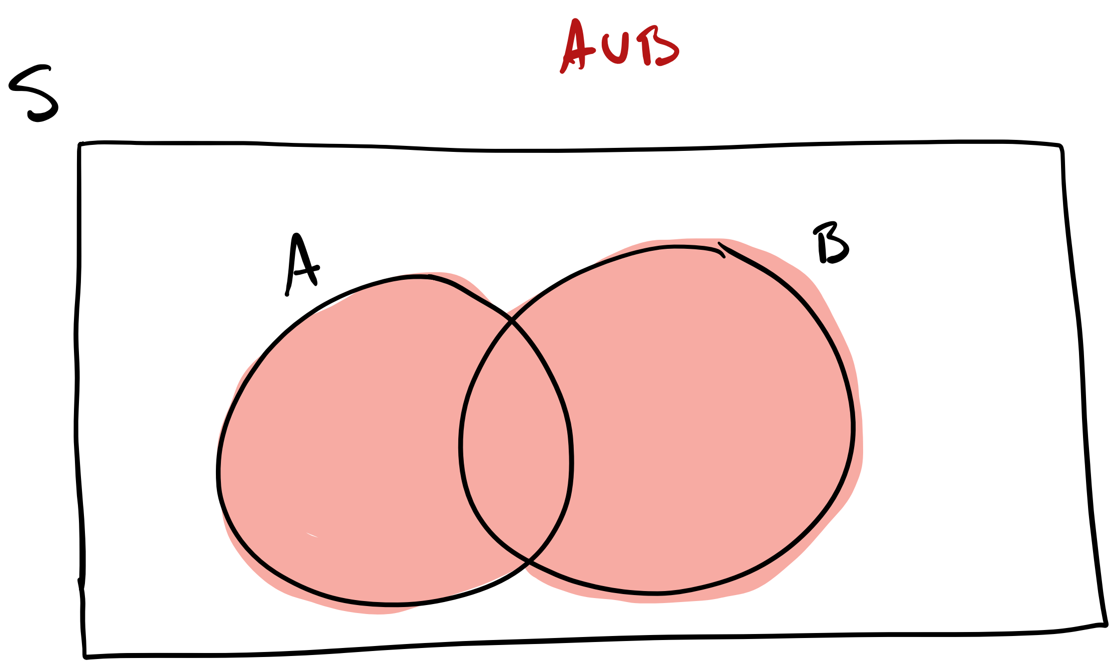
"set minus", takes away from S the elements in A.

$B = \{1, 2, 3\}$

Union $A \cup B = \{1, 2, 3, 5\}$

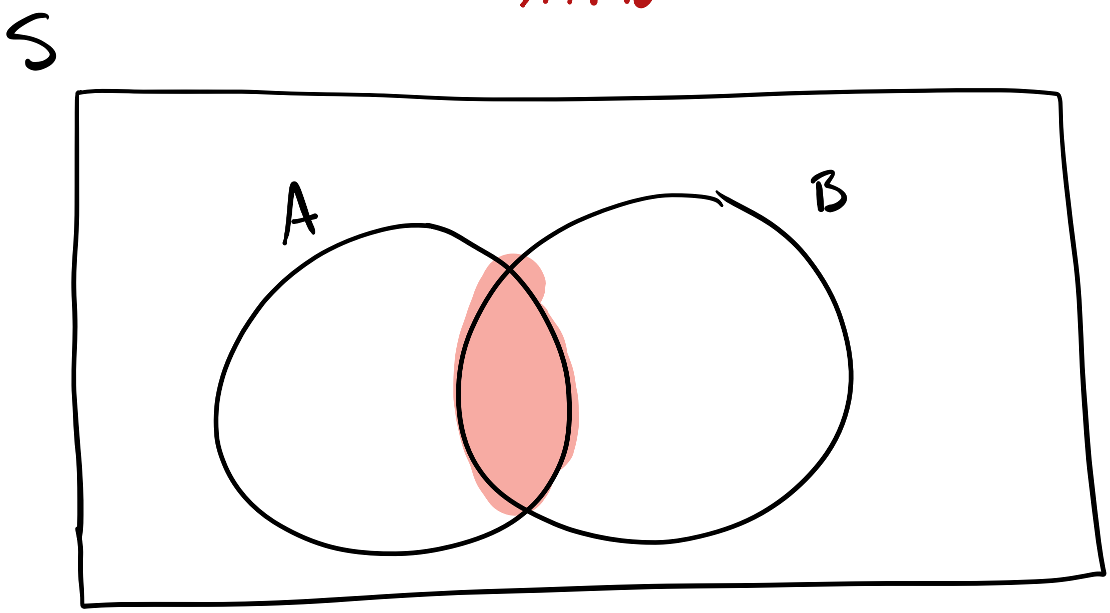
Intersection $A \cap B = \{1, 3\}$





Venn Diagram

$A \cap B$



Exercise: Suppose we flip a coin three times and record the sequence of heads and tails.

1. Give the sample space S.
2. Give the points in the event $A =$ at least two heads come up.
3. Give the points in the event $B =$ at least one tails comes up.
4. Give the points in the event $B \cap A$.
5. Give the points in the event $B^c \cup A^c$.
6. Give the points in the event $(B \cup A)^c$.
7. Give the points in the event $(B \cap A)^c$.
8. Give the points in A^c.

$B \cup A = S$

$S^c = S \setminus S$

$S = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}$

$$A = \{ HHH \quad HHT \quad HTH \quad THH \}$$

$$B = \{ HHT \quad HTH \quad THH \quad TTH \quad THT \quad HTT \quad TTT \}$$

$$B \cap A = \{ HHT \quad HTH \quad THH \}$$

$$B^c \cup A^c = \{HHH\} \cup \{ TTH \quad THT \quad HTT \quad TTT \}$$

$$= \{ HHH \quad TTH \quad THT \quad HTT \quad TTT \}$$

Empty set : $\phi = \{ \}$

The set which has no elements.

Exercise: Suppose we roll a 6-sided die twice. The sample space is

$$S = \left\{ \begin{matrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{matrix} \right\}.$$

B

Define the events

$A = \{\text{at least one of the rolls is odd}\}$

$B = \{\text{sum of the rolls is a prime number}\}$

$C = \{\text{sum of the rolls is an even number}\}$

$R = \{\text{sum of the rolls is } 7\}$

2, 3, 5, 7, 11

1. Give $A \cup B$.
2. Give $A \cap B$.
3. Give $B \cap C$.

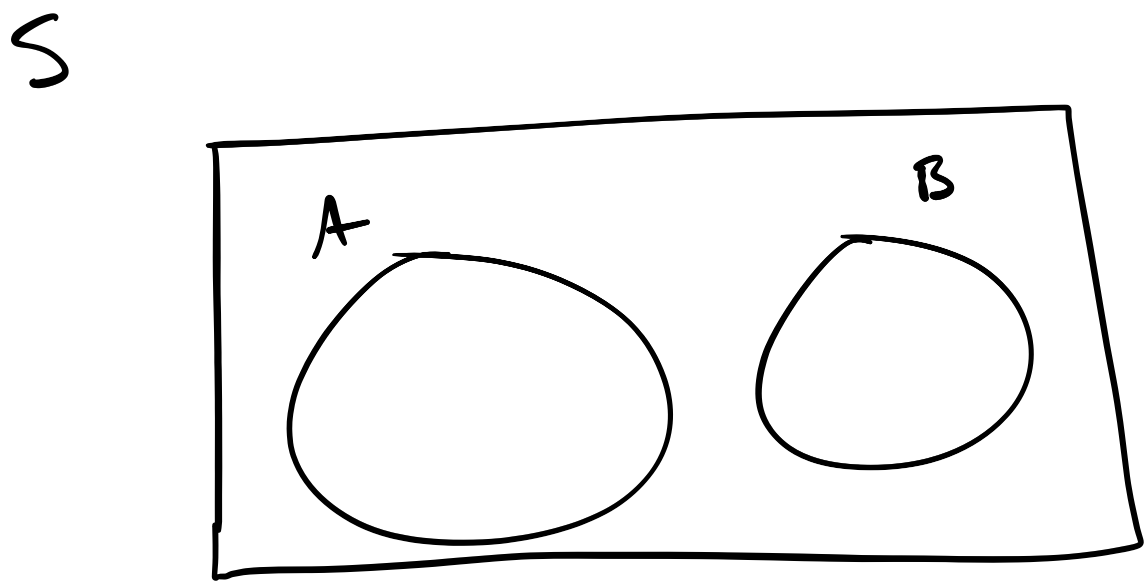
$B \cap C = \{(1,1)\}$

Mutual exclusivity/disjoint-ness

Two events A and B are called *mutually exclusive* or *disjoint* if $A \cap B = \emptyset$.

The set $\emptyset$ is the the *empty set*, which is the set containing no elements.

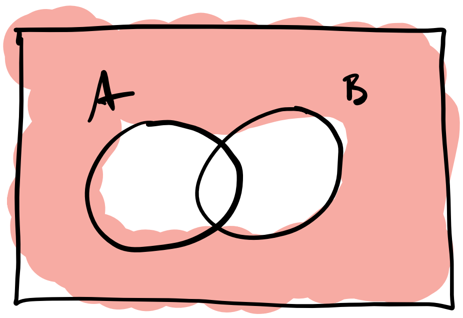
$A \cap B = \emptyset$



De Morgan:

$$(A \cup B)^c = A^c \cap B^c$$

S



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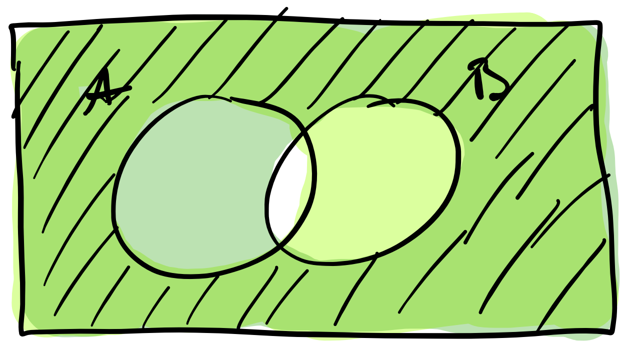

$$(A \cap B)^c = A^c \cup B^c$$

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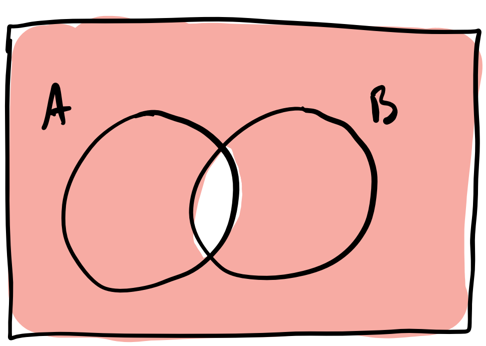

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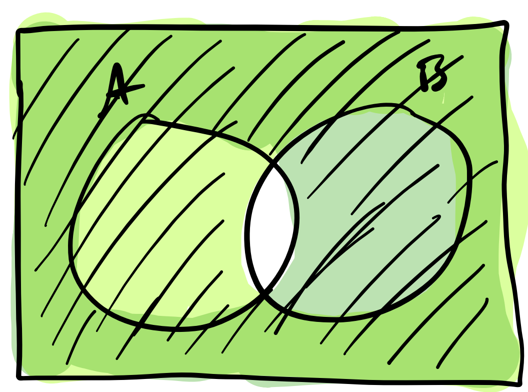


De Morgan's Laws

For any events A and B we have

$$(A \cup B)^c = A^c \cap B^c \quad \text{and} \quad (A \cap B)^c = A^c \cup B^c$$

Draw pictures.

Exercise: A birthday party might have cake C and it might have ice cream I. Interpret in words the events

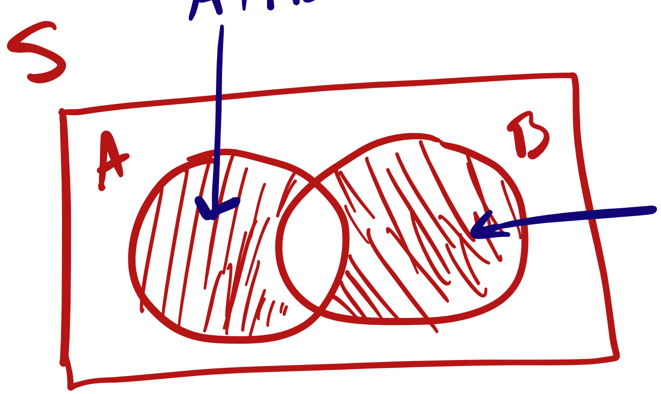
1. $(C \cap I)^c = C^c \cup I^c$ "Possibly no cake, possibly no ice cream, possibly no cake and no ice cream."
2. $(\underbrace{C \cup I}_{\text{"at least one of them"}})^c = C^c \cap I^c$ "No cake and no ice cream!"

Exercise: For two events A and B use our elementary set operations $\cup$, $\cap$, and the complement to give representations of the following:

- Both A and B occur. $A \cap B$
- Neither A nor B occurs; give two representations of this. $(A \cup B)^c = A^c \cap B^c$
- At least one of the events A and B occurs. $A \cup B$
- At least one of the events A and B does not occur; give two representations. $A^c \cup B^c = (A \cap B)^c$
- One of the events A and B occurs but not the other.

$(A \cap B^c) \cup (B \cap A^c)$

$A \cap B^c$



$B \cap A^c$

Exercise: Suppose a wildebeest W, a crocodile C, and a giraffe G on a safari. Write down the following events using elementary set operations. You see...

and

1. a giraffe ~~but~~ no wildebeest. $G \cap W^c$
2. all three types of animals. $W \cap C \cap G$
3. not all three types of animals. $(W \cap C \cap G)^c = W^c \cup C^c \cup G^c$
4. a giraffe and a wildebeest but no crocodile. $(G \cap W) \cap C^c$
5. not both a giraffe and a wildebeest.
6. a giraffe and a wildebeest without seeing a crocodile or a crocodile and a wildebeest without seeing a giraffe.
7. exactly two of the three types of animal.
8. exactly one of the three types of animal.
9. at least one of the three types of animal. $W \cup C \cup G$
10. none of the animals.

$$(W \cap (C \cup G)^c) \cup (C \cap (W \cup G)^c) \cup (G \cap (W \cup C)^c)$$

$A \cap B = \emptyset$ then A, B are "disjoint" or "mutually excl."

Partition

"mutually exclusive"

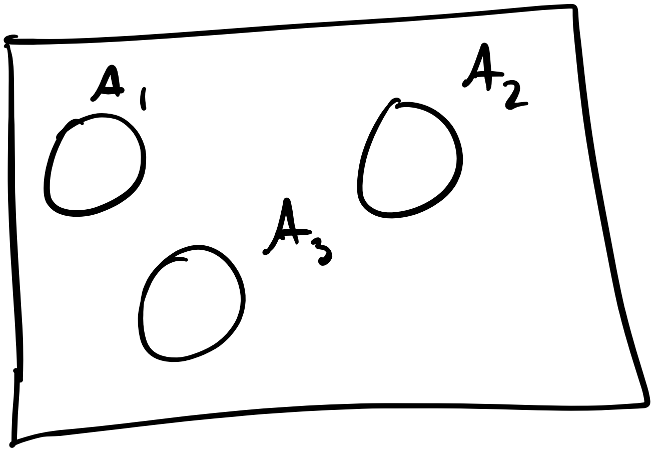
"

1. Events $A_1, A_2, \ldots$ are called *pairwise disjoint* if $A_i \cap A_j = \emptyset$ for all $i \neq j$.
2. If $A_1, A_2, \ldots$ are pairwise disjoint events in S and $A_1 \cup A_2 \cup \cdots = S$ then the collection of sets $A_1, A_2, \ldots$ is called a *partition* of S.

Draw pictures. . .

- Note that the two events A and A^c make up a partition of S.
- For a collection of sets, the terms "pairwise disjoint" and "mutually exclusive" can be used interchangeably.

S

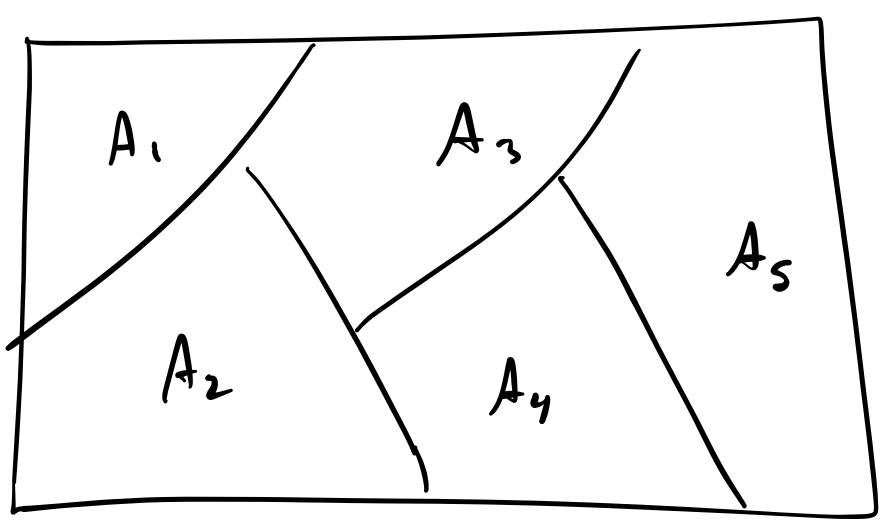


A_1, A_2, A_3 are pairwise disjoint.

$$A_1 \cap A_2 = \emptyset$$
$$A_2 \cap A_3 = \emptyset$$
$$A_1 \cap A_3 = \emptyset$$

$A_1, \dots, A_5$ form a partition of S.

S



Roll a 6-sided die: $S = \{1, 2, 3, 4, 5, 6\}$

$A_1 = \{1, 2\}$

$A_2 = \{3, 4\}$ $A_3 = \{5, 6\}$

pairwise disjoint and $A_1 \cup A_2 \cup A_3 = S$

1 Basics of sets

2 Basics of probability

Roll a 6-sided die:

$S = \{1, 2, 3, 4, 5, 6\}$

$A_1 = \{1, 2\}$

$A_2 = \{3, 4\}$

$A_3 = \{5, 6\}$

$P(A_1) = \frac{1}{3}$

$P(A_1 \cup A_2) = \frac{2}{3}$

$P(\underbrace{A_2 \cap A_3}_{\emptyset}) = 0$

$P(\underbrace{A_1 \cup A_2 \cup A_3}_{S}) = 1$

Introduce a "probability function" $P(\cdot)$

event

For an event A, we denote by $P(A)$ the probability that A occurs.

Examples:

1. Roll a die and let A be the event that ⚃ is rolled. Then $P(A) = 1/6$.
2. Roll two dice and let B be the event that ⚀, ⚀ is rolled. Then $P(B) = 1/36$.
3. Let C be the event that you get to park in your favorite spot. Then $P(C) =$ some number.

Now some axioms about probabilities. . .

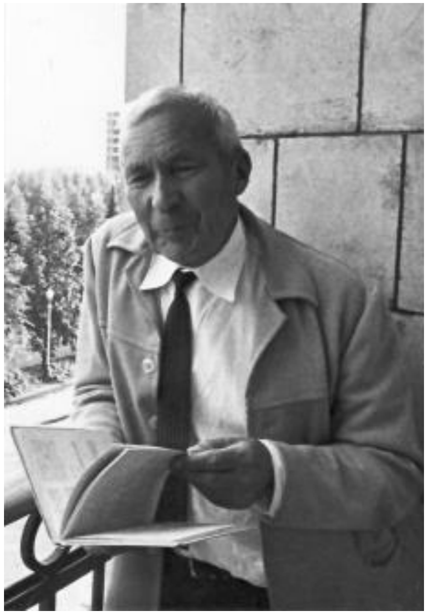
This is the first of the three Колмогоров axioms of probability.

3

Axiom (Nonnegativity of probabilities)

For any event A we must have $P(A) \geq 0$.

Here is $\boxed{\text{Колмогоров}}$ himself. Is he thinking about the axioms?



Kolmogorov

Axiom (Unity of the sample space probability)

For a statistical experiment with sample sample S, $P(S) = 1$.

Examples:

- Rolling a die: $S = \{$⚀, ⚁, ⚂, ⚃, ⚄, ⚅$\}$. We have $P(S) = 1$.
- MPG on next tank of gas: $S = [0, \infty)$. We have $P(S) = 1$.
- # freshman in class of 40: $S = \{0, 1, \ldots, 40\}$. $P(S) = 1$.

From die-rolling example: $S = \{1, 2, 3, 4, 5, 6\}$

$A_1 = \{1, 2\}$

$A_2 = \{3, 4\}$

$$P(A_1 \cup A_2) = \underbrace{P(A_1)}_{\frac{1}{3}} + \underbrace{P(A_2)}_{\frac{1}{3}} = \frac{2}{3}$$

Axiom (Probability of the union of mutually exclusive events)

If A and B are mutually exclusive events, then $P(A \cup B) = P(A) + P(B)$.

[1]

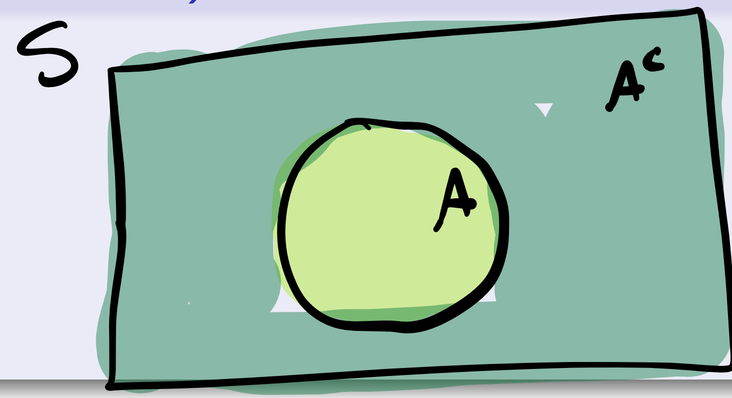
Example: Roll a die and let $A = \{⚀, ⚁\}$ and $B = \{⚂, ⚃\}$. What is $P(A \cup B)$?

[1] This is only a simplified version of the third Kolmogorov axiom.

Result (First consequences of Kolmogorov axioms)

Let A be an event in some sample space. Then

1. $P(\emptyset) = 0$
2. $P(A) \leq 1$
3. $\boxed{P(A^c) = 1 - P(A).}$ memorize



Examples:

- Roll a die and let $A = \{⚀, ⚁\}$. Then $P(A) = 1/3$. Give $P(A^c)$.
- We have $P(\emptyset) = 0$. Why?

Proof of 3: Write $S = A \cup A^c$.

Note that $A \cap A^c = \emptyset$, so

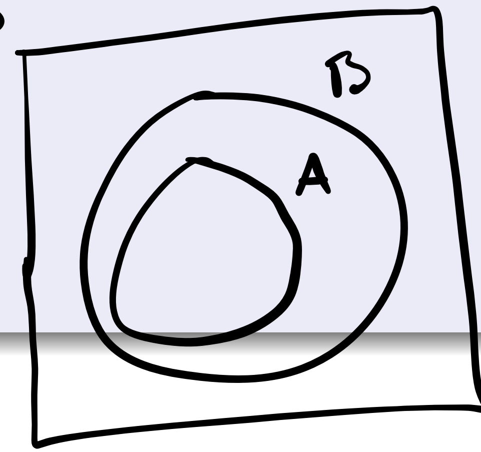
A, A^c are mutually exclusive.

This implies $\underbrace{P(\underbrace{A \cup A^c}_{S})}_{=1} = P(A) + P(A^c)$

Result (Further consequences of the Kolmogorov axioms)

Let A and B be any events in some sample space. Then:

S



1. $P(B \cap A^c) = P(B) - P(A \cap B)$
2. $\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B).}$ memorize
3. If $A \subset B$ then $P(A) \leq P(B)$.

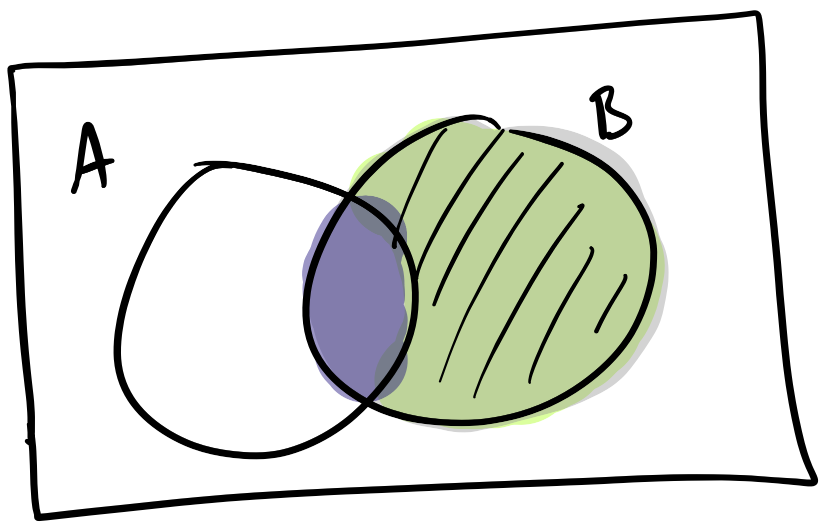
Draw pictures...

S

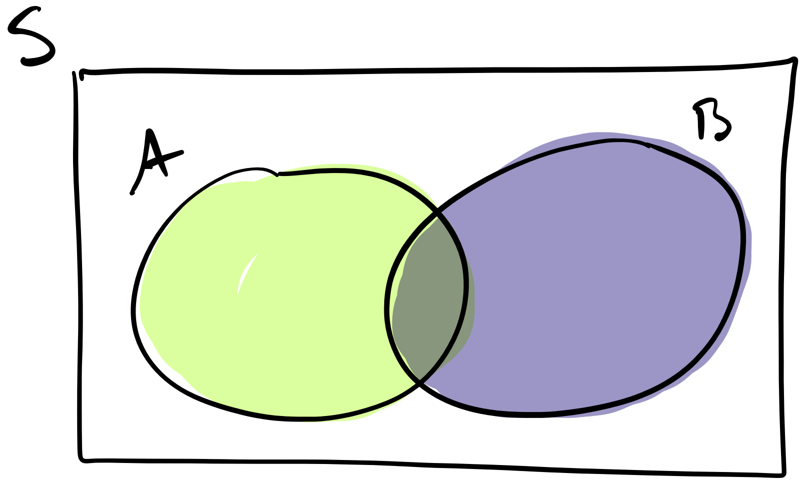
$P(B \cap A^c)$

$= P(B)$

$- P(A \cap B)$



$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



Exercise: Let F_1 and F_2 be the events that you meet friend 1 and friend 2 at a party, respectively. Suppose $P(F_1) = 0.4$, $P(F_2) = 0.5$ and $P(F_1 \cap F_2) = 0.01$.

Find the probabilities of the following:

1. You meet at least one of the two friends at a party.
2. You do not meet either of your friends at a party.
3. You meet friend 1 but not friend 2 at a party.

① $P(F_1 \cup F_2) = P(F_1) + P(F_2) - P(F_1 \cap F_2)$

$= 0.4 + 0.5 - 0.01$

$= 0.89$

(2) $P(F_1^c \cap F_2^c) = P((F_1 \cup F_2)^c)$

$= 1 - P(F_1 \cup F_2)$

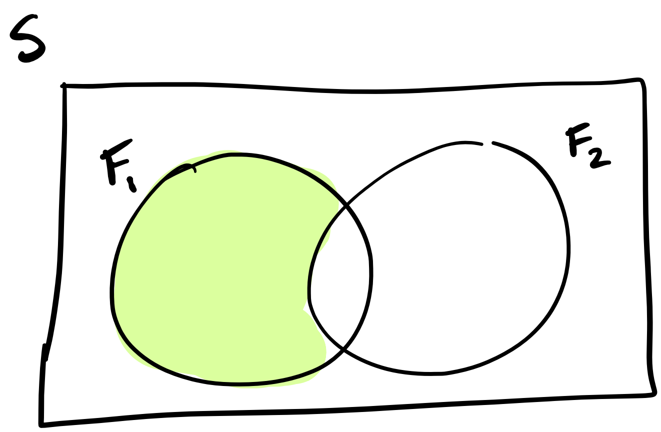
$= 1 - 0.89$

$= 0.11$

(3) $P(F_1 \cap F_2^c) = P(F_1) - P(F_1 \cap F_2)$

$= 0.4 - 0.01$

$= 0.39$



In some situations we can compute probabilities by counting sample points.

Computing probabilities when all sample points equally likely

If all outcomes in S are equally likely, for any event $A \subset S$, we have

$$P(A) = \frac{\#\{\text{sample points in } A\}}{\#\{\text{sample points in } S\}}.$$

Exercise: Suppose we roll two 6-sided dice. Find

1. P(sum of rolls equals 7)
2. P(we roll doubles)
3. P(sum of rolls at least 10)
4. P(at least one roll is greater than 3)

Exercise: A jukebox will play a song at random from a library of 50 songs, of which 20 are R&B, 10 are country, 12 are pop, and 8 are rock. Of the R&B songs, 15 are less than a decade old, and of each other genre, half of the songs are less than a decade old. Find the probability that the jukebox plays

1. A pop song.
2. An R&B song less than a decade old.
3. A song less than a decade old.
4. A rock song more than a decade old.
5. A country song or any song that is more than a decade old.

Result (Law of total probability)

Let A be an event in S and let $C_1, C_2, \dots$ be a partition of S. Then

$$P(A) = P(A \cap C_1) + P(A \cap C_2) + \dots$$

Can be useful to write A as the union of several mutually exclusive sets.