STAT 515 Lec 02 slides

Counting rules

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These slides are an instructional aid; their sole purpose is to display, during the lecture, definitions, plots, results, etc. which take too much time to write by hand on the blackboard. They are not intended to explain or expound on any material.

Motivation to study counting rules

If all outcomes in S are equally likely, for any event $A \subset S$, we have

$$P(A) = \frac{\#\{\text{sample points in } A\}}{\#\{\text{sample points in } S\}}.$$

This leads to an interest in counting rules.

Fundamental theorem of counting:

A job requires K tasks,

The tasks can be done in

$$n_1, n_2, \ldots, n_K$$

different ways.

Then total # ways to do the job is

$$\prod_{i=1}^{K} n_i = n_1 \times n_2 \times \cdots \times n_K .$$

Fundamental theorem of counting

If a job consists of K tasks such that the tasks may be completed in $n_1, \dots, n_K$ ways, respectively, then there are

$$\prod_{k=1}^{K} n_k = n_1 \times n_2 \times \cdots \times n_K$$

ways to do the job.

Exercise: You are confronted with the following sequence of choices:

1. Barbacoa, chicken, carnitas, or veggies
2. White or brown rice
3. Black or pinto beans
4. Spicy, medium, or mild
5. To pay extra for guacamole or not to pay extra for guacamole

In how many ways can you build your burrito?

$K = 5$

$n_1 = 4$

$n_2 = 2$

$n_3 = 2$

$n_4 = 3$

$n_5 = 2$

$$\# \text{ways} = \underbrace{4 \times 2}_{8} \cdot \underbrace{2 \cdot \overbrace{3 \cdot 2}^{6}}_{12}$$

$$= 96$$

$$\frac{N!}{(N-r)!} = \frac{N \cdot (N-1) \cdot (N-2) \cdots (N-r+1)\,\cancel{(N-r)}\,\cancel{(N-r-1)} \cdots 1}{\cancel{(N-r)} \cdot \cancel{(N-r-1)} \cdot \cancel{(N-r-2)} \cdots 1}$$

$$= N(N-1) \cdot \cdots \cdot (N-r+1)$$

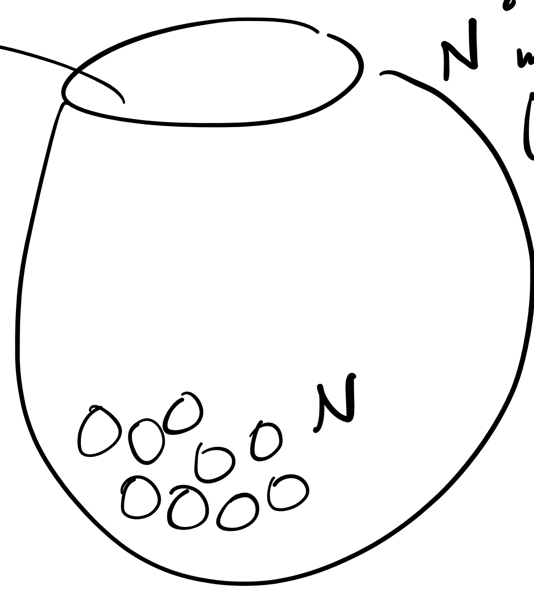
$$N \times (N-1) \times (N-2) \times \cdots \times (N-r+1)$$

$$\underbrace{o \cdots o}_{r}$$

bag of

N marbles

(every marble unique)



$5! = 5 \times 4 \times 3 \times 2 \times 1$

"five factorial"

Permutation

Draw r elements of N elements without replacement and arrange them in some order. The # ways is

$$\boxed{N(N-1)\cdots(N-r+1)} = \boxed{\frac{N!}{(N-r)!}}.$$

Exercise: You must compose a ballad in the key of G, i.e. from the chords

G Am Bm C D Em ~~F#dim~~

$N = 7$

$r = 4$

1. In how many ways can four distinct chords be chosen to begin your song?
2. What if you can't play F#dim?

$N - r + 1 = 4$

(1) $7 \cdot 6 \cdot 5 \cdot 4 = 42 \cdot 20 = 840$

$$\frac{7!}{(7-4)!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1} = 7 \cdot 6 \cdot 5 \cdot 4 = 840$$

(2) $N = 6$

$$\frac{6!}{(6-4)!} = \frac{6!}{2!} = 6 \cdot 5 \cdot 4 \cdot 3 = 360$$

$0\,0\,0\,0\,0\,0$

Exercise: Jane Austen wrote the following novels:

$$\frac{N!}{(N-r)!} = \frac{6!}{(6-6)!} = \frac{6!}{0!} = 6! = \underbrace{6\cdot5\cdot4\cdot3\cdot2\cdot1}_{30\times12\times2} = 720$$

- Pride and Prejudice
- Emma
- Sense and Sensibility
- Persuasion
- Northanger Abbey
- Mansfield Park

$4!$

$0! \equiv 1$

000
000

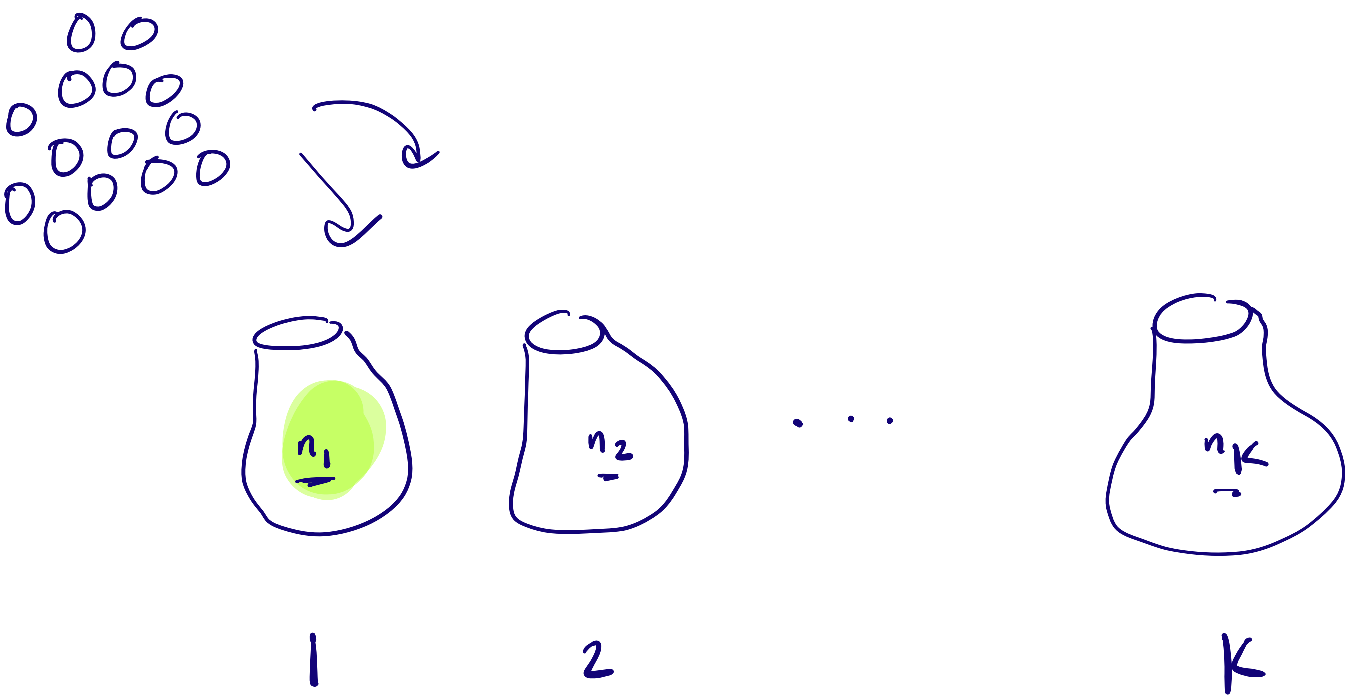
$\boxed{r=N}$

$N=6$
$r=6$

1. In how many ways can you read them all this ~~semester~~ year?
2. In how many ways can you read them all such that you read Pride and Prejudice, Emma, and Sense and Sensibility without reading any of the others in between?

$$4! * 3! = 4\cdot3\cdot2\cdot1 \times 3\cdot2\cdot1 = 4\cdot6\cdot6 = 144.$$

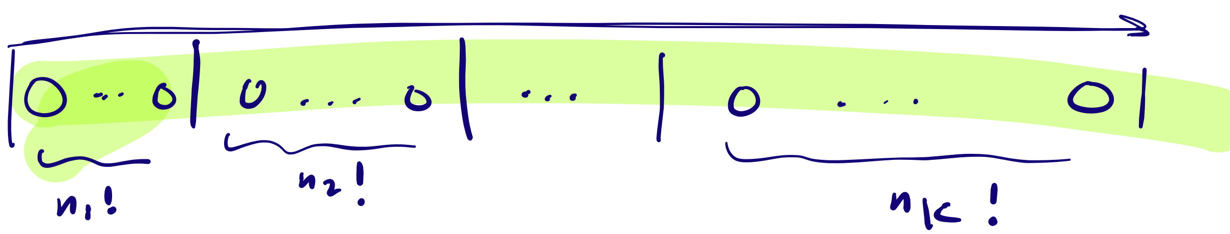
N balls



K urns

Like this:

1. Line up the N balls



N! ways

$$\# \text{ ways} = \frac{N!}{n_1! \cdot n_2! \cdot \ldots \cdot n_K!}$$

Partition

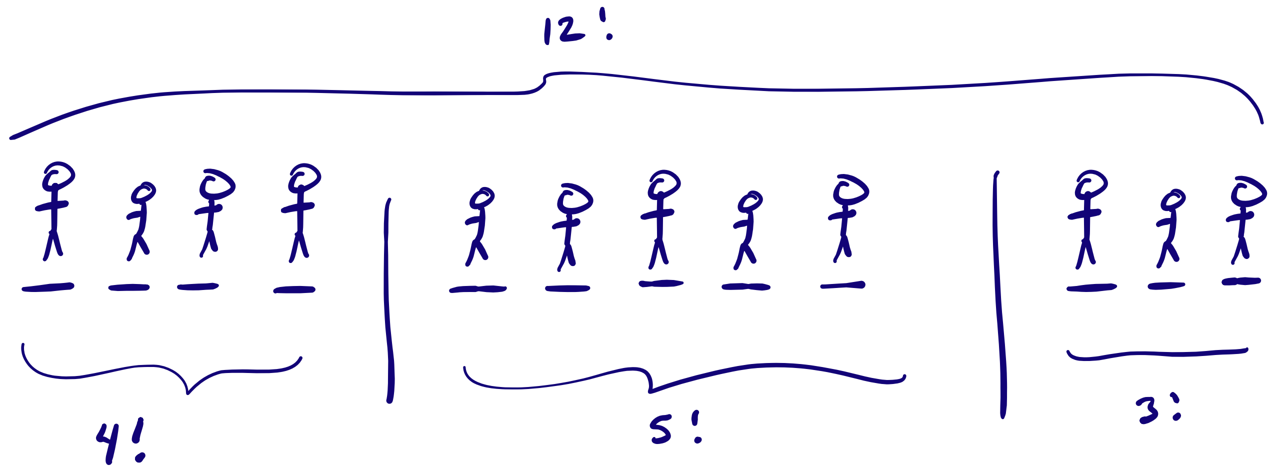
Place N balls into K urns such that the K urns receive $n_1, \dots, n_K$ balls, where $N = n_1 + \cdots + n_N$. The # ways is

$$\frac{N!}{n_1!n_2!\cdots n_K!}.$$

Exercise: Suppose 12 people are randomly assigned to ride in 3 vehicles taking 4, 5, and 3 passengers, respectively.

1. In how many ways can the passengers be assigned to the different vehicles?
2. If you and your friend are among these people, with what probability will the two of you ride in the same vehicle?

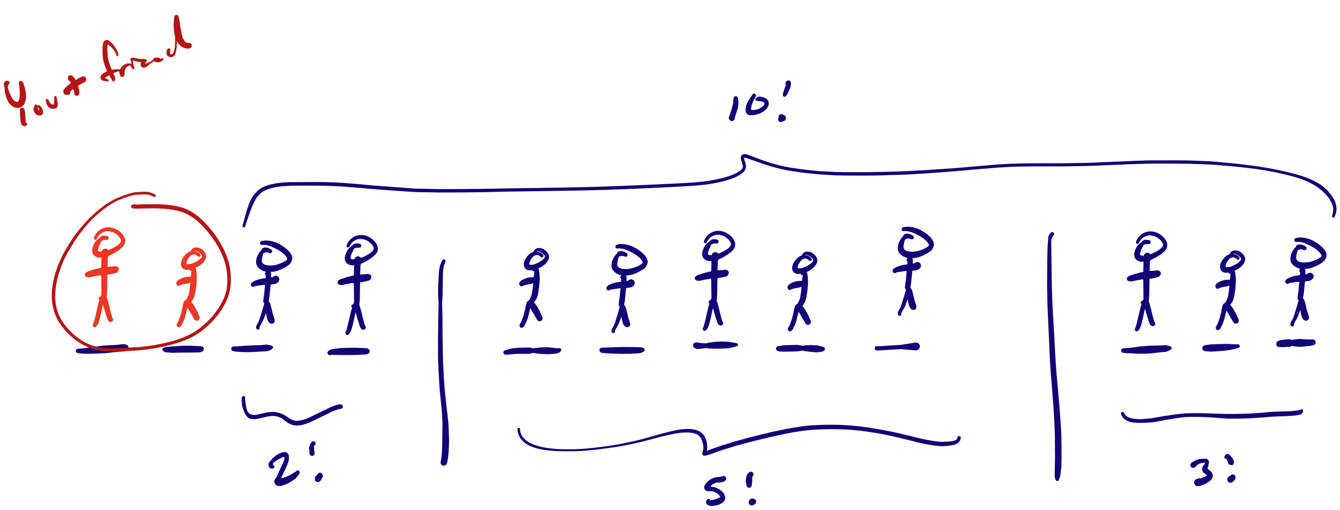
①



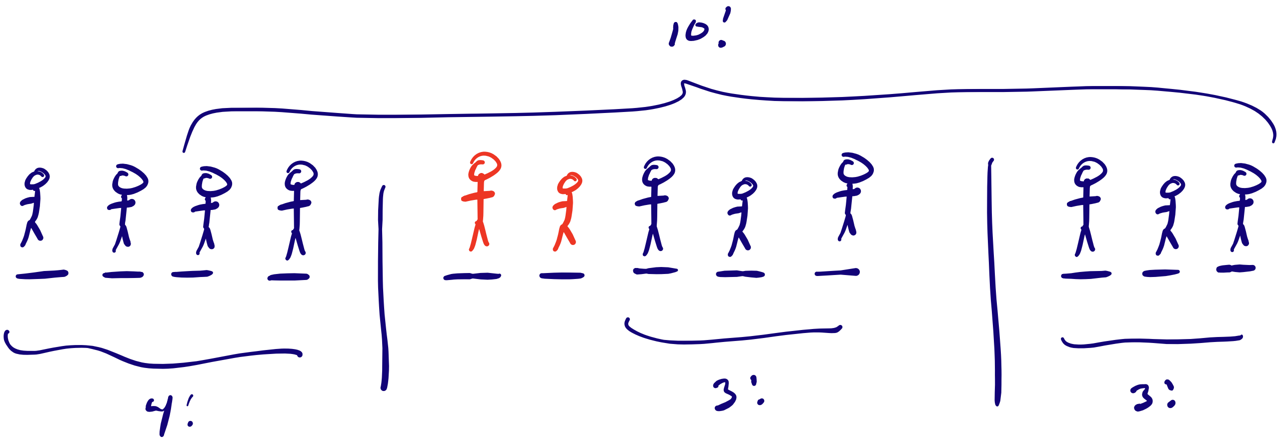
$$= \frac{12!}{4!\ 5!\ 3!} = 27{,}720$$

②

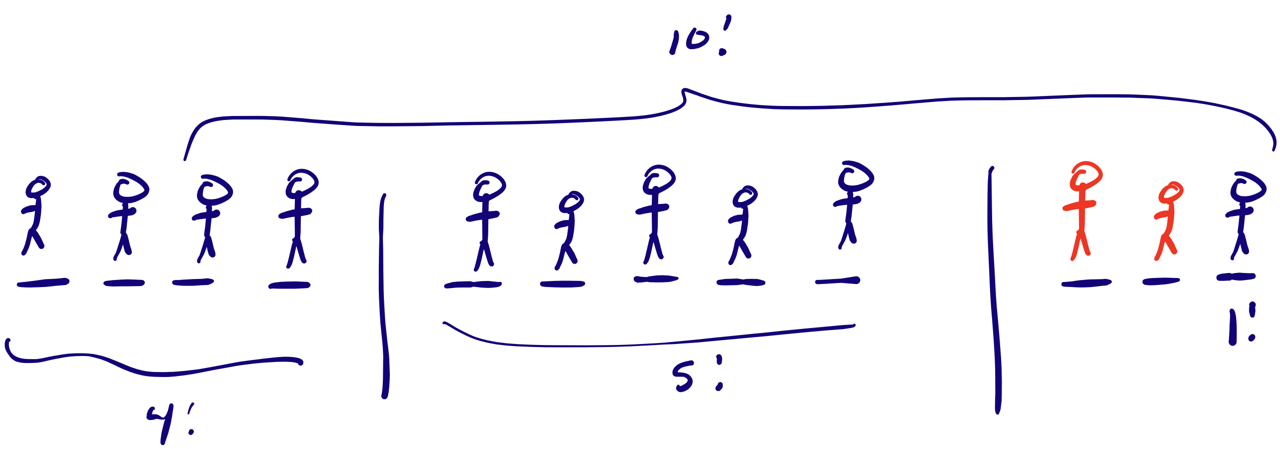
$$P\left(\text{You are in a vehicle with your friend}\right) = \frac{\#\{\text{You with friend}\}}{\#\{\text{total ways}\}}$$



$$\frac{10!}{2!\,5!\,3!} = 2,520$$



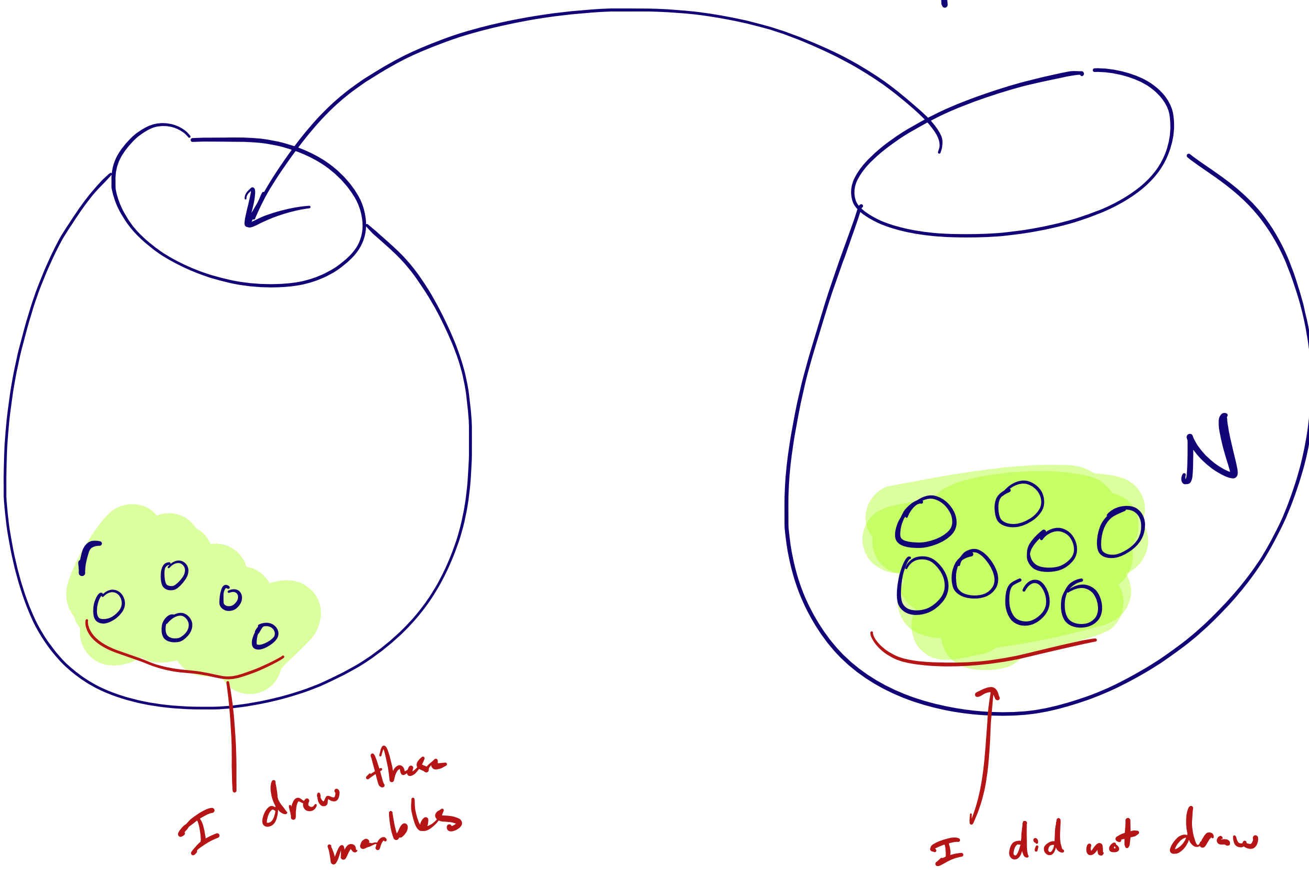
$$\frac{10!}{4!\,3!\,3!} = 4200$$



$$\frac{10!}{4!\,5!\,1!} = 1260$$

$$\text{Prob} = \frac{2520 + 4200 + 1260}{27720} = 0.288.$$

Draw r marbles from bag of N marbles,
put in another bag.



$$\# \text{ ways} = \frac{N!}{r!\,(N-r)!}$$

"Combination"

"N choose r"

Combination

Draw r elements from N without replacement and without regard to their order. The # ways is

$$\binom{N}{r} = \frac{N!}{r!(N-r)!}.$$

This is just a partition with only 2 "urns".

Exercise: In how many ways could you choose 2 of the 6 J.A. novels to read?

$N = 6$

$r = 2$

$$\frac{6!}{2!(6-2)!} = \frac{6!}{2!\,4!} = \frac{6 \cdot 5}{2} = 15$$

Exercise: For German class you must watch 5 episodes from among 10 available episodes of Bares für Rares, 8 of Betty's Diagnose, and 12 of Haustier Check.

1. In how many ways can you choose
   a) 5 episodes of Bares für Rares?
   b) 3 episodes of $\boxed{\text{Haustier Check}}$ and 2 episodes of $\boxed{\text{Betty's Diagnose?}}$
2. If you choose 5 episodes at random, with what probability do you
   a) not watch any episodes of Haustier Check?
   b) watch at least one episode of Bares für Rares?
   c) binge entirely on Betty's Diagnose?

10 BR, 8 BD, 12 HC

Select 5 episodes

① (a)

$N = 10$

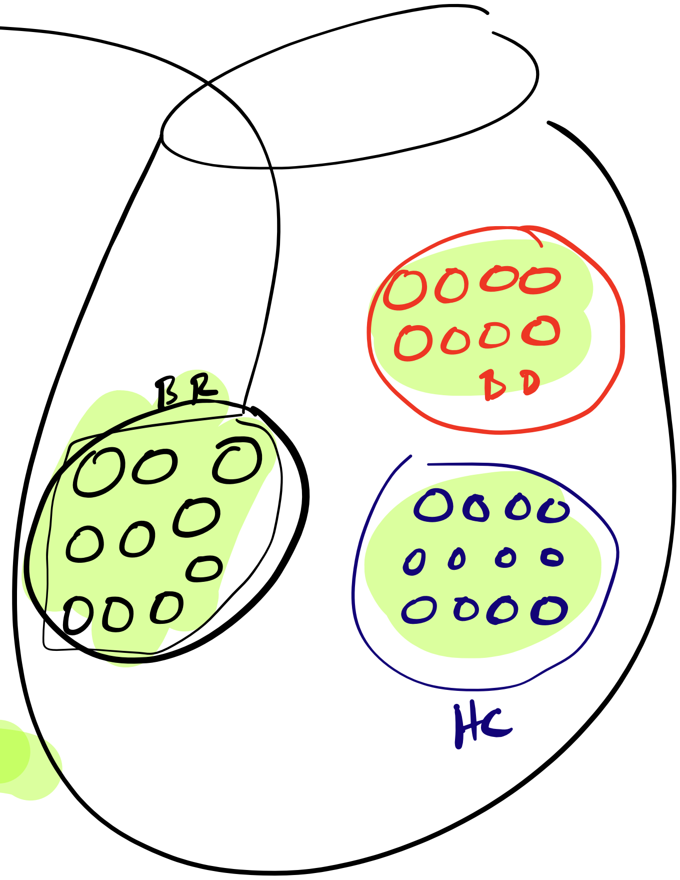
$r = 5$

$\# \text{ ways} = \binom{10}{5} = \frac{10!}{5!(10-5)!}$

$= \frac{{}_{2}\cancel{10} \cdot 9 \cdot \cancel{8}^{2} \cdot 7 \cdot \cancel{6}}{\cancel{5} \cdot \cancel{4} \cdot \cancel{3 \cdot 2}}$

$= 2 \cdot 9 \cdot 2 \cdot 7$

$= 252$



(b) $\underbrace{\text{3 from HC}}_{\binom{12}{3} \text{ ways}}$, $\underbrace{\text{2 from BD.}}_{\binom{8}{2}}$

$\# \text{ ways} \quad \binom{12}{3} \cdot \binom{8}{2} = \frac{12!}{3!\,9!} \quad \frac{8!}{2!\,6!}$

$= \frac{12 \cdot 11 \cdot 10}{3 \cdot 2} \cdot \frac{8 \cdot 7}{2}$

$= 6{,}160$

2. If you choose 5 episodes at random, with what probability do you
   a) not watch any episodes of Haustier Check?
   b) watch at least one episode of Bares für Rares?
   c) binge entirely on Betty's Diagnose?

(2) (a)

$$P(A) = \frac{\#\{\text{points in } A\}}{\#\{\text{points in sample space}\}}$$

$$\#\{\text{points in sample space}\} = \binom{30}{5} = 142{,}506$$

$$\#\{\text{ways no HC episodes}\} = \binom{18}{5} = 8568$$

$$P(\text{no HC episodes}) = \frac{\binom{18}{5}}{\binom{30}{5}} = \frac{8568}{142{,}506} = 0.0601.$$

(b) $P(\text{At least 1 from BR})$

$$= 1 - P(\text{None from BR})$$

$$= 1 - \frac{\#\{\text{None from BR}\}}{\binom{30}{5}}$$

$$= 1 - \frac{\binom{20}{5}}{\binom{30}{5}}$$

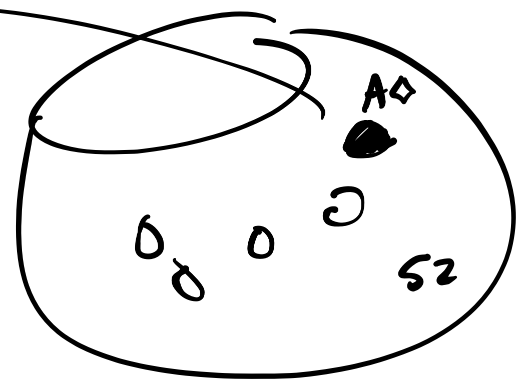
$$= 0.891$$

(c) $$P(\text{All 5 from BD}) = \frac{\#\{\text{ways 5 from BD}\}}{\binom{30}{5}}$$

$$= \frac{\binom{8}{5}}{\binom{30}{5}}$$

$$= 0.00039$$

$1 * \binom{51}{4}$



Exercise: If dealt 5 cards from a 52-card deck, what is the probability of getting

1. the ace of diamonds?
2. at least one ace?

$$P(A\diamond) = \frac{\#\{\text{hands w/ } A\diamond\}}{\#\{5\text{-card hands}\}} = \frac{1 * \binom{51}{4}}{\binom{52}{5}}$$

$$= \frac{\dfrac{51!}{4!\,\cancel{47!}}}{\dfrac{52!}{5!\,\cancel{47!}}}$$

$$= \frac{\frac{51!}{4!}}{\frac{52!}{5!}}$$

$$= \frac{51!}{4!} \cdot \frac{5!}{52!}$$

$$= \frac{5}{52}$$

② P(At least one Ace)

$$= 1 - P(\text{No Ace})$$

$$= 1 - \frac{\binom{48}{5}}{\binom{52}{5}}$$

$$= 0.341$$