# STAT 515 Lec 03 slides

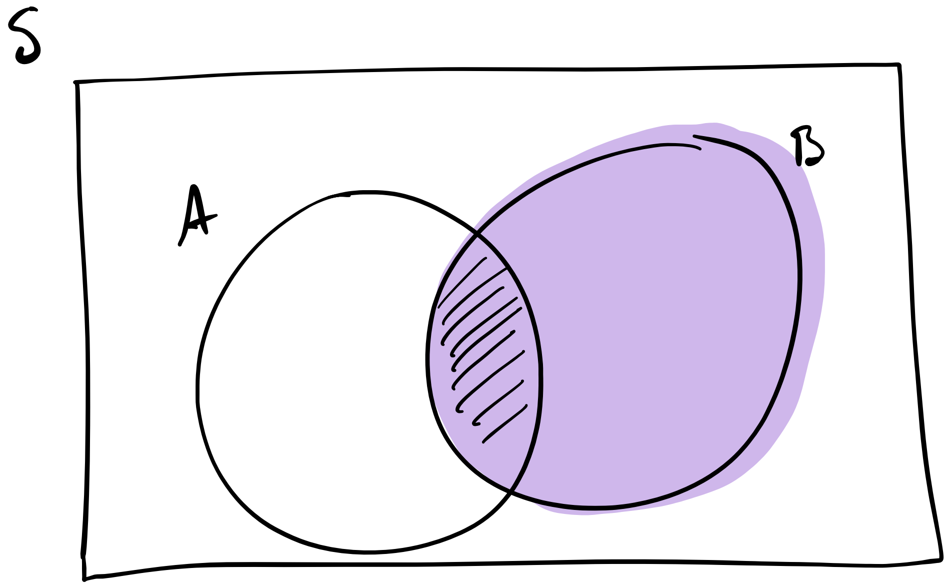
# Conditional probability, independence, Bayes' rule

Karl Gregory

University of South Carolina

These slides are an instructional aid; their sole purpose is to display, during the lecture, definitions, plots, results, etc. which take too much time to write by hand on the blackboard. They are not intended to explain or expound on any material.

# Conditional probability




Conditional probability of A given B is

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

conditioning event

## Conditional probability

The *conditional probability* of $A$ given $B$ is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

This is the probability that the event $A$ occurs given that $B$ occurs.

**Exercise:** Roll two dice. Find

1. $P($ doubles $)$
2. $P($ sum $\geq 10)$
3. $P($ doubles $|$ sum $\geq 10)$
4. $P($ sum $\geq 10|$ doubles $)$

$P(\text{Doubles}) = 6/36 = \frac{1}{6}$

$P(\text{Sum} \geq 10) = 6/36 = 1/6$

$$\mathcal{S} = \left\{ \begin{array}{cccccc} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{array} \right\}$$

$P(\text{Doubles} \mid \text{Sum} \geq 10) = \dfrac{P(\text{Doubles} \cap \text{Sum} \geq 10)}{P(\text{Sum} \geq 10)} = \dfrac{2/36}{6/36} = \dfrac{1}{3}$

$P(\text{Sum} \geq 10 \mid \text{Doubles}) = \dfrac{2/36}{6/36} = \dfrac{1}{3}$

**Exercise:** From STAT 515 fa 2019:

- 40 students in class
- 10 students got an ‘A’ on final exam
- 12 students got an ‘A‘ hw average
- 23 students did not get an ‘A‘ on the final or an ‘A’ hw average.

If a student is drawn at random from the class, give

1. $P$( ‘A’ on final )
2. $P$( ‘A’ on final | ‘A‘ hw average )
3. $P$( ‘A’ on final | less than ‘A‘ hw average )
4. $P$( less than ‘A’ on final | less than ‘A‘ hw average )
5. $P$(‘A’ on final $\cap$ ‘A‘ hw average )

| | A final | $(\text{A final})^c$ | total |
|---|---|---|---|
| A hw | 5 | 7 | 12 |
| $(\text{A hw})^c$ | 5 | 23 | 28 |
| total | 10 | 30 | 40 |

<u>Experiment:</u> Select student at random.

$$P(\text{A final}) = \frac{10}{40}$$

$$P(\text{A final} \mid \text{A hw}) = \frac{P(\text{A final} \cap \text{A hw})}{P(\text{A hw})}$$

$$= \frac{5/40}{12/40}$$

$$= \frac{5}{12} = 0.417$$

$$P(\text{A final} \mid (\text{A hw})^c) = \frac{P(\text{A final} \cap (\text{A hw})^c)}{P((\text{A hw})^c)}$$

$$= \frac{5/40}{28/40}$$

$$= \frac{5}{28} = 0.179$$

**Rule:**

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$\Longleftrightarrow \quad P(A \cap B) = P(A|B)\,P(B)$$

Intersection = conditional × unconditional

$$P(B|A) = \frac{P(A \cap B)}{P(B)} \quad \Longleftrightarrow \quad P(A \cap B) = P(B|A)\,P(A)$$

## Intersection prob. as conditional times unconditional

For any two events $A$ and $B$,

$$P(A \cap B) = P(A|B)P(B) \quad \text{or} \quad P(A \cap B) = P(B|A)P(A).$$

**Exercise:** Suppose that on a safari, the probabilities of seeing a giraffe ($G$), a wildebeest ($W$), and a crocodile ($C$) are as follows:

$P(W) = 0.40$

$P(C) = 0.60$

$P(G) = 0.20$

$P(C|W) = 0.775$

$P(C|G) = 0.65$

$P(G \cap W) = 0.06$

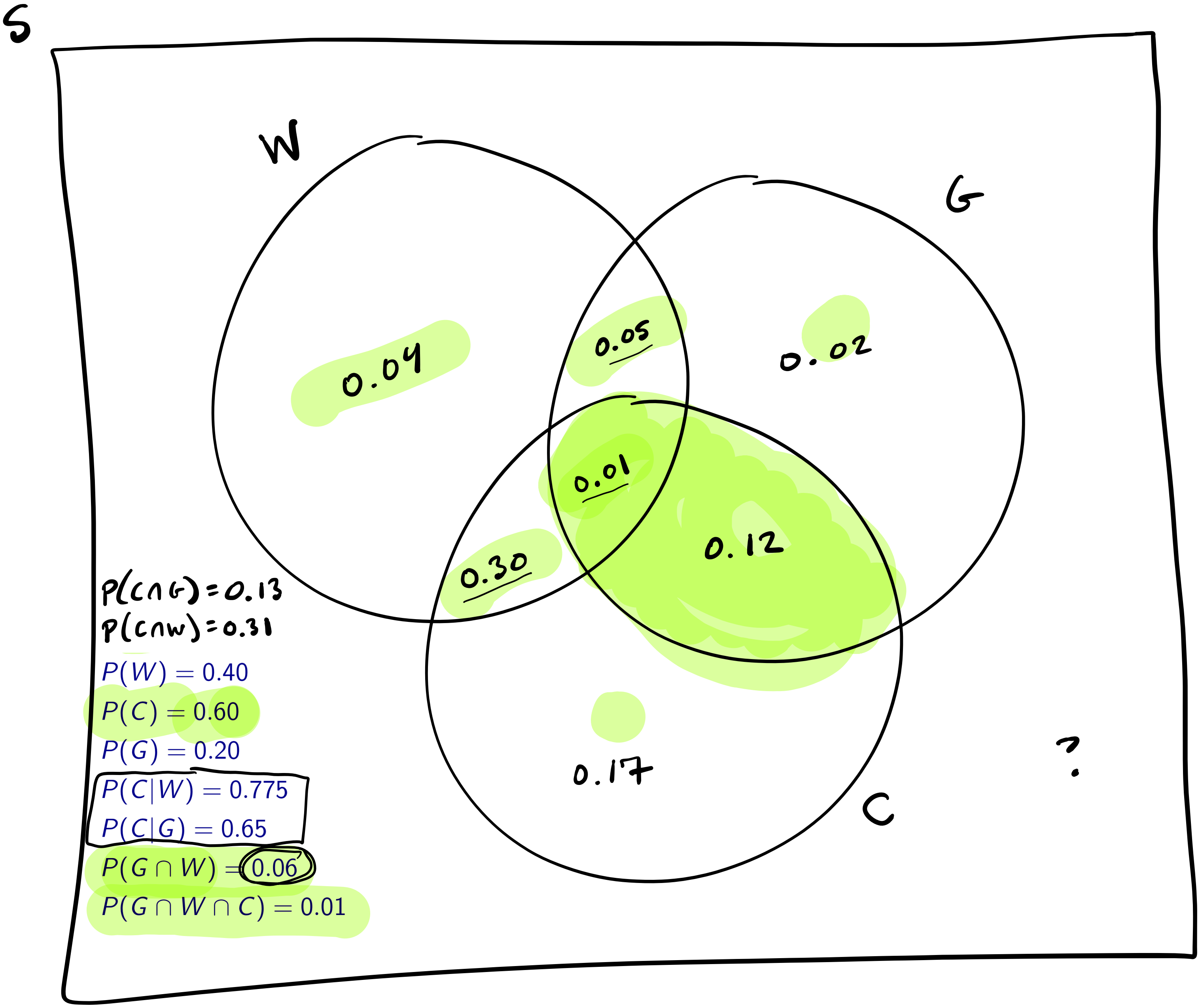
$P(G \cap W \cap C) = 0.01$

$$P(C|W) = \frac{P(C \cap W)}{P(W)} \Rightarrow$$

$$P(C \cap W) = \underbrace{P(C|W)}_{0.775}\underbrace{P(W)}_{0.40} = 0.31$$

$P(C \cap G) = P(C|G)P(G) = 0.65 * 0.2 = 0.13$

Fill out a Venn diagram with the probabilities of all possibilities.




$P(C \cap G) = 0.13$

$P(C \cap W) = 0.31$

$P(W) = 0.40$

$P(C) = 0.60$

$P(G) = 0.20$

$P(C|W) = 0.775$

$P(C|G) = 0.65$

$P(G \cap W) = 0.06$

$P(G \cap W \cap C) = 0.01$

## Independence

Two events $A$ and $B$ are called *independent* if

$$P(A \cap B) = P(A)P(B).$$

## Equivalent definitions of independence

The following statements are equivalent:

- $P(A \cap B) = P(A)P(B)$
- $\boxed{P(A) = P(A|B)}$
- $P(B) = P(B|A)$

$A, B$ independent $\Rightarrow$ $P(A|B) = \frac{P(A \cap B)}{P(B)}$

$= \frac{P(A)\cancel{P(B)}}{\cancel{P(B)}}$

$= P(A)$

Also: If $A, B$ independent, so are the pairs of events $A, B^c$ and $A^c, B$ and $A^c, B^c$.

Prob. of $A$ is unaffected by occurrence of $B$.

**Exercise:** Flip a coin twice and let

$$H_1 = \text{heads on first flip}$$
$$H_2 = \text{heads on second flip}$$

Find $P(H_1 \cap H_2)$ assuming that the flips are independent.

$$P(H_1 \cap H_2) = P(H_1)\,P(H_2) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

**Exercise:** Let

- $A =$ flat tire
- $B =$ forgot spare tube

Suppose that $P(A) = 0.02$ and $P(B) = 0.10$ and $P(A \cap B) = 0.002.$

Are the events independent? $P(A \cap B) = P(A)P(B)$

$P(A)P(B) = 0.02 \times 0.10 = 0.002 = P(A \cap B).$

Yes, independent.

A, B indep ⟹
$P(A \cap B) = P(A)\,P(B)$

**Exercise:** Send survey to 10 people. Let $R_i$ = person $i$ responds, $i = 1, \ldots, 10$. Assume independence with probability of response 0.20. Give

1. $P($ Everyone completes survey $)$
2. $P($ No one completes survey $)$
3. $P($ At least one person completes the survey $)$

$R_1$: person 1 responds
⋮
$R_{10}$: person 10 responds

1. $P(R_1 \cap R_2 \cap \cdots \cap R_{10}) = P(R_1)\,P(R_2) \cdot \ldots \cdot P(R_{10})$

$$= \underbrace{(0.20)(0.20) \cdot \ldots \cdot (0.20)}_{10}$$

$$= (0.20)^{10} \quad (\text{tiny})$$

② $P(\text{No one completes}) = P(R_1^c \cap R_2^c \cap \cdots \cap R_{10}^c)$

$$= P(R_1^c)P(R_2^c) \cdot \ldots \cdot P(R_{10}^c)$$

$$= (0.80)^{10} = 0.11$$

③ $P(\text{At least 1 completes}) = 1 - P(\text{No one})$

$$= 1 - (0.80)^{10}$$

$$= 1 - 0.11$$

$$= 0.89.$$

At least $= P(R_1 \cup R_2 \cup \cdots \cup R_{10})$

$$= 1 - P((R_1 \cup R_2 \cup \cdots \cup R_{10})^c)$$

$$= 1 - \underbrace{P(R_1^c \cap R_2^c \cap \cdots \cap R_{10}^c)}_{(0.80)^{10}}$$

**Exercise:** From STAT 515 fa 2019:

- 40 students in class
- 10 students got an ‘A’ on final exam
- 12 students got an ‘A‘ hw average
- 23 students did not get an ‘A‘ on the final or an ‘A’ hw average.

If a student is drawn at random from the class, are the events ‘A’ on final and ‘A’ hw average independent?

| | A final | (A final)$^c$ | total |
|---|---|---|---|
| A hw | 5 | 7 | 12 |
| (A hw)$^c$ | 5 | 23 | 28 |
| total | 10 | 30 | 40 |

Q: Draw a student at random from among these 40.

Events "A on final" and "A hw avg" independent?

[Independence: $P(A \cap B) = P(A)\,P(B)$]

$$P(\text{"A on final"})\,P(\text{"A hw avg"}) = \frac{10}{40} \cdot \frac{12}{40} = \frac{120}{1600} = \frac{12}{160} = \frac{3}{40}$$

$$P(\text{"A on final"} \cap \text{"A hw avg"}) = \frac{5}{40}$$

NOT EQUAL

## Bayes' Rule $\boxed{\text{(simplified)}}$

For any two events $A$ and $B$ such that $P(A) > 0$,

$$P(B|A) = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|B^c)P(B^c)}.$$

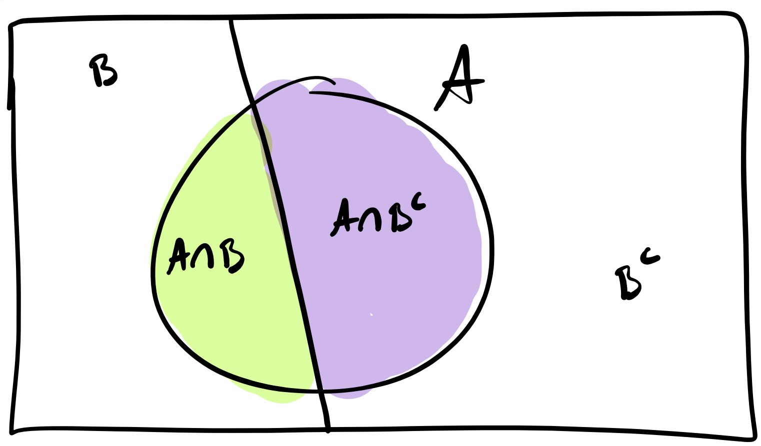
Show why...

$P(A|B) = \frac{P(A\cap B)}{P(B)}$ $\quad$ $P(B|A) = \frac{P(A\cap B)}{P(A)}$

**Bayes' Rule:**

$$P(B|A) = \frac{P(A\cap B)}{P(A)} = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|B^c)P(B^c)}$$

S




$$P(A) = P(A\cap B) + P(A\cap B^c)$$
$$= P(A|B)P(B) + P(A|B^c)P(B^c)$$

**Exercise:** Suppose 20% of a population has an infection. Suppose a test is positive with probability 0.70 when the infection is present and negative with probability 0.95 when the infection is absent.

$I$ : "infection"

$T$ : "test positive"

$$P(I) = 0.20$$

$$P(T \mid I) = 0.70$$ "Sensitivity"

$$P(T^c \mid I^c) = 0.95$$ "Specificity"

$$\boxed{P(A^c \mid B) = 1 - P(A \mid B)}$$

Complement rule works with conditional probs.

$$P(I \mid T) = \frac{P(T \mid I)\, P(I)}{P(T \mid I)\, P(I) + \underbrace{P(T \mid I^c)}_{1-0.95 = 0.05}\, \underbrace{P(I^c)}_{1-P(I) = 0.80}}$$

$$= \frac{(0.70)(0.20)}{(0.70)(0.20) + (0.05)(0.80)}$$

$$= 0.778$$

$$P(I^c \mid T^c) = \frac{P(T^c \mid I^c)\, P(I^c)}{P(T^c \mid I^c)\, P(I^c) + \underbrace{P(T^c \mid I)}_{1 - P(T \mid I) = 1 - 0.70 = 0.30}\, P(I)}$$

$$= \frac{(0.95)(0.80)}{(0.95)(0.80) + (0.30)(0.20)}$$

$$= 0.927$$

Get two tests. $T_1$, $T_2$ are "positive" results for test 1 and 2, respectively.

Assume tests don't affect each other.

$$P(I \mid \underbrace{T_1 \cap T_2}_{\text{both test positive}}) = \frac{\overbrace{P(T_1 \cap T_2 \mid I)}^{P(T_1 \mid I) \cdot P(T_2 \mid I) = (0.7)(0.7)} \; \overbrace{P(I)}^{=(0.2)}}{P(T_1 \cap T_2 \mid I)\, P(I) + \underbrace{P(T_1 \cap T_2 \mid I^c)}_{(0.05)(0.05)} \underbrace{P(I^c)}_{(0.80)}}$$

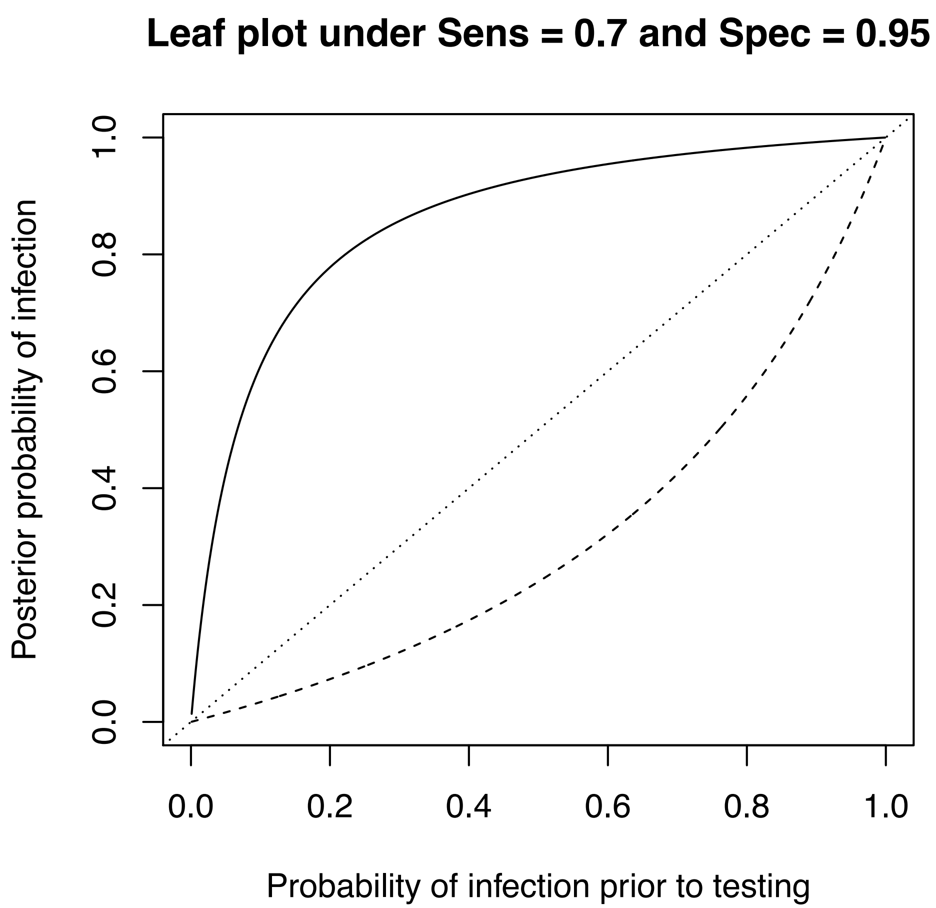
$$= \frac{(0.7)(0.7)(0.2)}{(0.7)(0.7)(0.2) + (0.05)(0.05)(0.80)}$$

$$= 0.98$$

**Exercise:** Suppose 20% of a population has an infection. Suppose a test is positive with probability 0.70 when the infection is present and negative with probability 0.95 when the infection is absent.

For a randomly selected person from the population, find:

1. $P($ infection $|$ positive test $)$
2. $P($ no infection $|$ negative test $)$
3. If 100 people are tested, among whom 20 have the infection, how many do you expect of
   - False positives
   - True positives
   - False negatives
   - True negatives
4. Suppose a person is tested twice, with test outcomes independent. Find
   - $P($ infection $|$ two positive tests$)$
   - $P($ infection $|$ two negative tests$)$


Leaf plot under Sens = 0.7 and Spec = 0.95
Posterior probability of infection
0.0
0.2
0.4
0.6
0.8
1.0
0.0
0.2
0.4
0.6
0.8
1.0
Probability of infection prior to testing

## Bayes' Rule

For an event $A$ in $S$ and a partition $B_1, \ldots, B_K$ of $S$, we have

$$P(B_k|A) = \frac{P(A|B_k)P(B_k)}{P(A|B_1)P(B_1) + \cdots + P(A|B_K)P(B_K)}.$$

Show why...

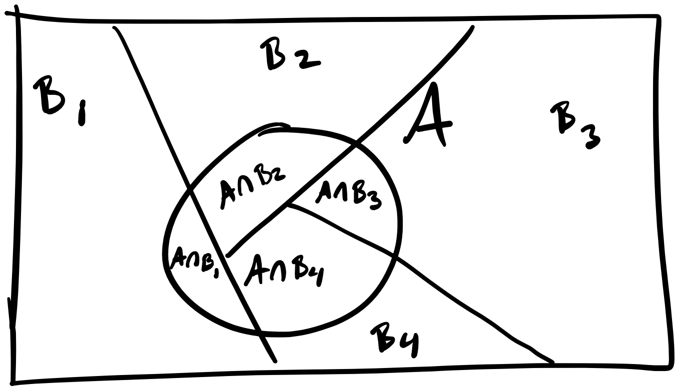
$B_1 = B \quad B_2 = B^c, \quad K = 2.$

Get simplified version noting that $B$ and $B^c$ form a partition of $S$.

$$P(B_k \mid A) = \frac{P(A \mid B_k)\,P(B_k)}{P(A \mid B_1)P(B_1) + \cdots + P(A \mid B_k)P(B_k)}$$

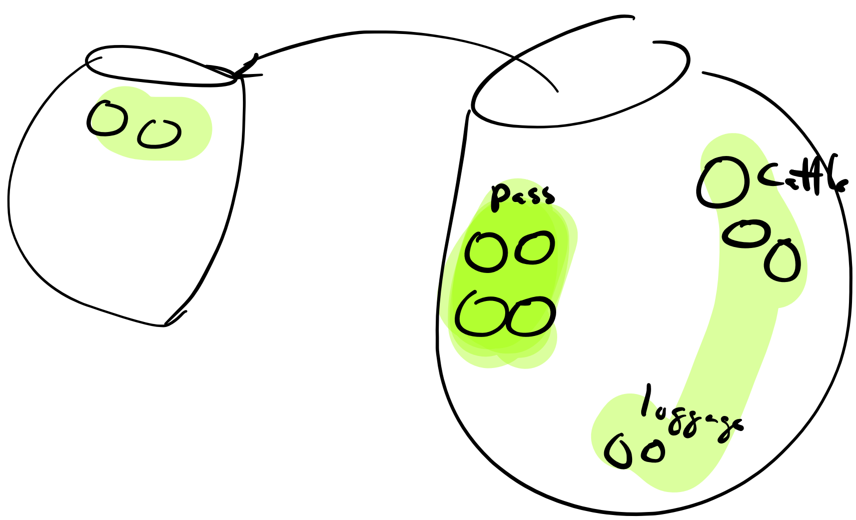
$$P(A) = P(A \cap B_1) + \cdots + P(A \cap B_k)$$
$$= P(A \mid B_1)P(B_1) + \cdots + P(A \mid B_k)P(B_k)$$

S




HW 2

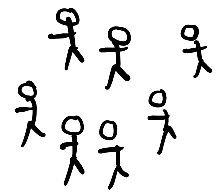
3 (a) (i) $P(\text{Two passenger}) = \dfrac{\binom{4}{2}}{\binom{9}{2}}$




(b) "Jane Austin Books"

(c) Partition: Look at partition example.

(d) (i) $7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 7!$



(e) $\underbrace{2 \cdot 2 \cdot 2 \cdots \; 2}_{2^{10}}$

4 Magic bowling ball: $P(M) = 1/5$

$P(S|M) = 3/4$

$P(S|M^c) = 1/4$

(b) $$P(M|S) = \frac{P(S|M)\,P(M)}{P(S|M)P(M) + P(S|M^c)\,P(M^c)}$$

(c) Look at "two positive tests example"

**Exercise:** Roll a die and draw one bill from a bag as follows:

| | | | |
|---|---|---|---|
| Roll ⚀, ⚁, ⚂ | $\longrightarrow$ | draw from bag 1: | 3 \$5 bills and 1 \$20 bill |
| Roll ⚃, ⚄ | $\longrightarrow$ | draw from bag 2: | 2 \$5 bills and 2 \$20 bills |
| Roll ⚅ | $\longrightarrow$ | draw from bag 3: | 1 \$5 bill and 3 \$20 bills |

$B_1, B_2, B_3$ are events

1. What is the probability that you get \$20?
2. Given that you get \$20, what is the probability that you drew from bag 1?
3. If you did this 1000 times:
   - How many times would you expect to get \$20?
   - Of the times you get \$20, on how many do you expect it to be from bag 1?

$$P(\$20) = P(\$20 \cap B_1) + P(\$20 \cap B_2) + P(\$20 \cap B_3)$$
$$= P(\$20 \mid B_1)P(B_1) + P(\$20 \mid B_2)P(B_2) + P(\$20 \mid B_3)P(B_3)$$

$$= \frac{1}{4} \cdot \frac{1}{2} + \frac{2}{4} \cdot \frac{1}{3} + \frac{3}{4} \cdot \frac{1}{6}$$

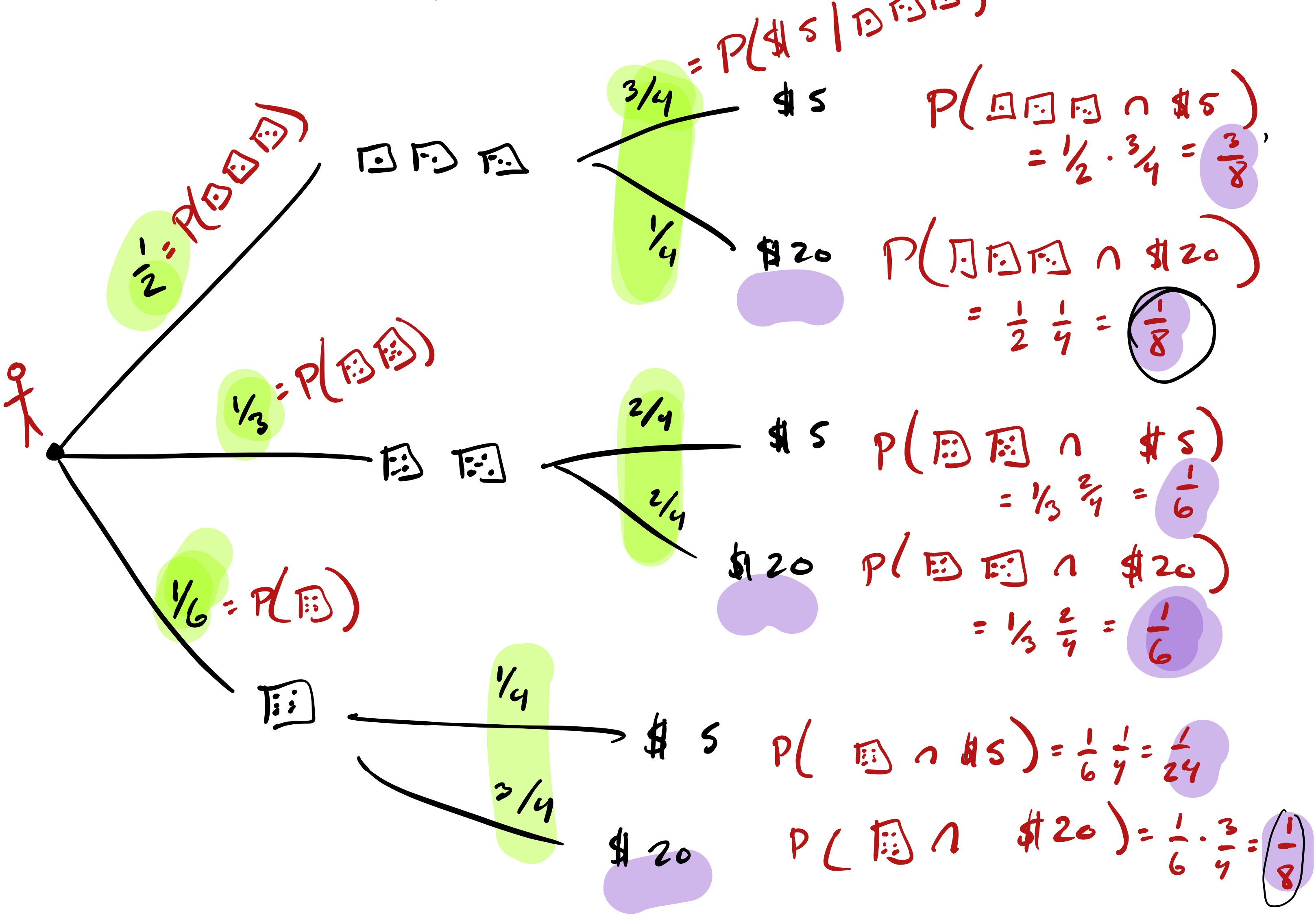
$$= \frac{1}{8} + \frac{2}{12} + \frac{3}{24}$$

$$= \frac{3 + 4 + 3}{24}$$

$$= \frac{10}{24}$$

$$= \frac{5}{12}$$

## TREE DIAGRAM



① $$P(\$20) = \frac{1}{8} + \frac{1}{6} + \frac{1}{8} = \frac{3+4+3}{24} = \frac{10}{24}$$

$$= \frac{5}{12}.$$

1. What is the probability that you get $20?
2. Given that you get $20, what is the probability that you drew from bag 1?

$$P(B_1 \mid \$20) = \frac{P(B_1 \cap \$20)}{P(\$20)} = \frac{1/8}{5/12}$$

$$= \frac{1}{8}\,\frac{12}{5}$$

$$= \frac{6}{4.5}$$

$$= \frac{6}{20}$$

$$= \frac{3}{10}$$

$$P(B_2 \mid \$20) = \frac{P(B_2 \cap \$20)}{P(\$20)}$$

$$= \frac{1/6}{5/12}$$

$$= \frac{12}{30}$$

$$= \frac{4}{10}$$

$$P(B_3 \mid \$20) = \frac{P(B_3 \cap \$20)}{P(\$20)} = \frac{1/8}{5/12} = \frac{3}{10}.$$