STAT 515 Lec 08 slides

The Poisson and Exponential distributions

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Suppose X discrete

with support $\mathcal{X} = \{0, 1, 2, \dots\}$

These slides are an instructional aid; their sole purpose is to display, during the lecture, definitions, plots, results, etc. which take too much time to write by hand on the blackboard. They are not intended to explain or expound on any material.

Poisson process

Suppose $X = \#$ occurrences per unit of time or space, where the occurrences

1. are independent
2. occur randomly but at a constant rate over the entire time/space.

A process generating such occurrences is called a *Poisson process*.

Examples:

1. # customers entering a store in an hour.
2. # earthquakes per decade in a region.
3. # weeds growing per square meter of a field.
4. # bird nests per acre in a habitat.

Poisson distribution

The probability distribution with pmf given by

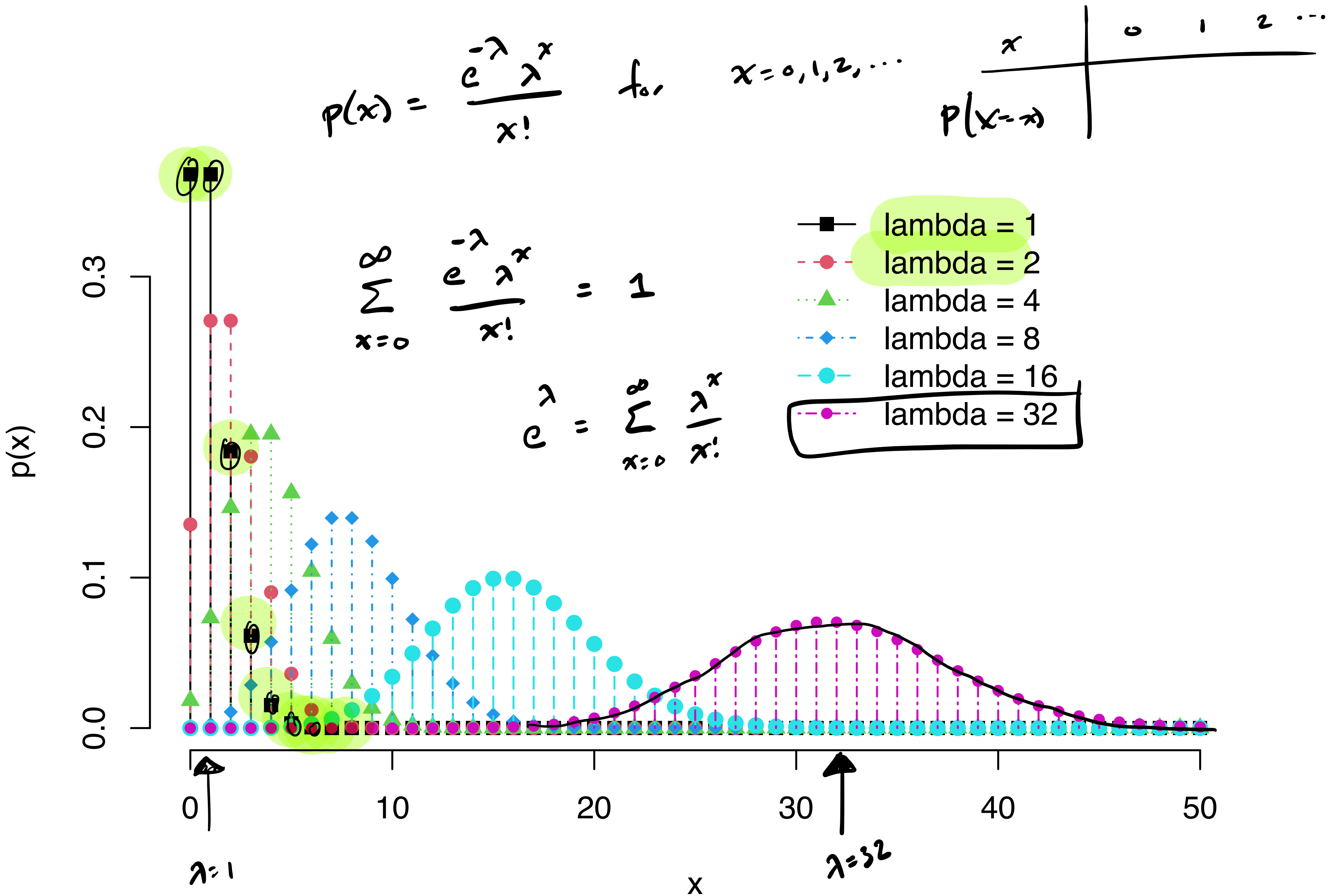
Probability mass function

$$P(X=x) = p(x) = \frac{e^{-\lambda}\lambda^x}{x!} \quad \text{for } x = 0, 1, \ldots$$

with $\lambda > 0$ is called the *Poisson distribution*.

λ: lambda (Λ)

We write $X \sim \text{Poisson}(\lambda)$.



$p(x) = \frac{e^{-\lambda}\lambda^x}{x!}$ for $x = 0, 1, 2, \ldots$
x | 0 1 2 ...
p(X=x)
$\sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^x}{x!} = 1$
$e^{\lambda} = \sum_{x=0}^{\infty} \frac{\lambda^x}{x!}$
lambda = 1
lambda = 2
lambda = 4
lambda = 8
lambda = 16
lambda = 32
p(x)
0.0
0.1
0.2
0.3
0
10
20
30
40
50
x
λ=1
λ=32

Mean and variance of Poisson distribution

If $X \sim \text{Poisson}(\lambda)$ then

- $\mathbb{E}X = \lambda$.
- $\text{Var}\, X = \lambda$.

$\lambda = 20$

Exercise: Let $X \sim \text{Poisson}(20)$ be the # car accidents in town on a given day.

1. Find $P(X = 10)$.
2. Find the probability that there is at least one accident.
3. Find $P(X \leq 15)$.
4. Find $P(X \geq 20)$.
5. If X observed on many days, to what will the average of the values be close?

Introduce `dpois` and `ppois` functions in R.

$X \sim \text{Poisson}(20)$.

$$p(x) = \frac{e^{-\lambda}\lambda^{x}}{x!}$$

① $$P(X=10) = \frac{e^{-20}\,20^{10}}{10!} = 0.0058 = \text{dpois}(10, 20)$$

← On exam stop here.

(10 ↑ x, 20 ↑ λ)

② $$P(X \geq 1) = 1 - P(X < 1)$$

$$= 1 - P(X=0)$$

$$= 1 - \frac{e^{-20}\,20^{0}}{0!}$$

$$= 1 - e^{-20}$$

$$= 1 - \frac{1}{e^{20}}$$

$$\approx 1$$

③ $$P(X \leq 15) = P(X=0) + P(X=1) + \cdots + P(X=15)$$

$$= \sum_{x=0}^{15} P(X=x)$$

$$= \sum_{x=0}^{15} \frac{e^{-20}\,20^{x}}{x!}$$

← If on exam, stop here

$$= \text{ppois}(15, 20)$$

(20 ↑ λ)

$= 0.1565$

$$= \sum_{x=20}^{\infty} \frac{e^{-20}\, 20^{x}}{x!}$$

on exam leave like this.

(4) $P(X \geq 20) = 1 - P(X < 20)$

$$= 1 - P(X \leq 19)$$

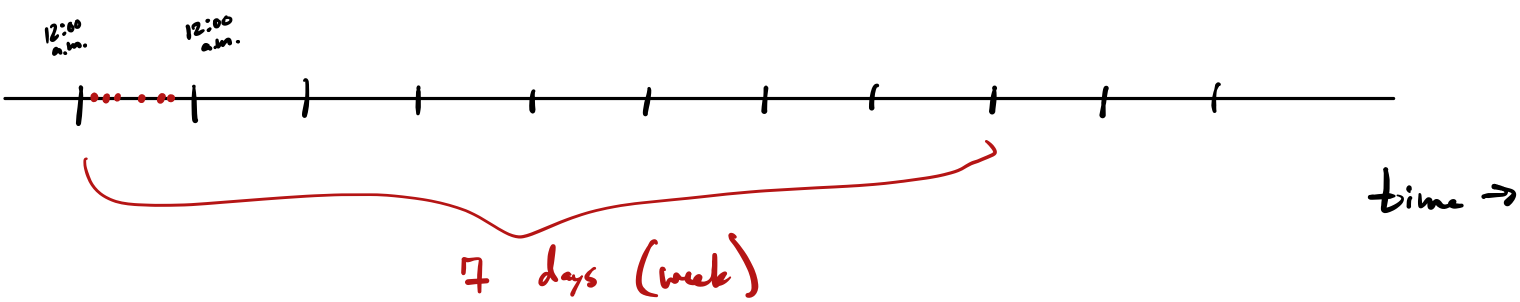
$$= 1 - \left[\sum_{x=0}^{19} \frac{e^{-20}\, 20^{x}}{x!} \right]$$

$$= 1 - \text{ppois}(19, 20)$$

$$= 0.5297$$

X = # car accidents in 1 day $\sim$ Poisson(λ)

X_7 = # car accidents in 7 days $\sim$ Poisson$(7 \cdot \lambda)$



Mean number of occurrences scales with unit of time/space. . .

Let $X \sim \text{Poisson}(\lambda)$, where $X = \#$ occurrences per unit time/space of an event.

Then if $X_t = \#$ occurrences in t units of time/space, we have $X_t \sim \text{Poisson}(t\lambda)$.

Exercise: Let X_7 be the $\#$ car accidents in town in a given week.

1. What is the distribution of X_7? Refer to previous example.
2. Find $P(X_7 \leq 130)$.
3. Find $P(X_7 = 140)$.
4. Find $P(X_7 \geq 150)$.

① $X_7 \sim \text{Poisson}(7 \cdot 20 = 140)$ $\qquad X \sim \text{Poisson}(\lambda = 20)$

② $P(X_7 \leq 130) = \sum_{x=0}^{130} P(X_7 = x)$

$= \sum_{x=0}^{130} \frac{e^{-140}\, 140^x}{x!}$

$= \text{ppois}(130, 140)$

↑ x ↑ λ

$= 0.2124.$

③ $P(X_7 = 140) = \frac{e^{-140}\, 140^{140}}{140!} = \text{dpois}(140, 140)$

↑ x ↑ λ

$= 0.0337$

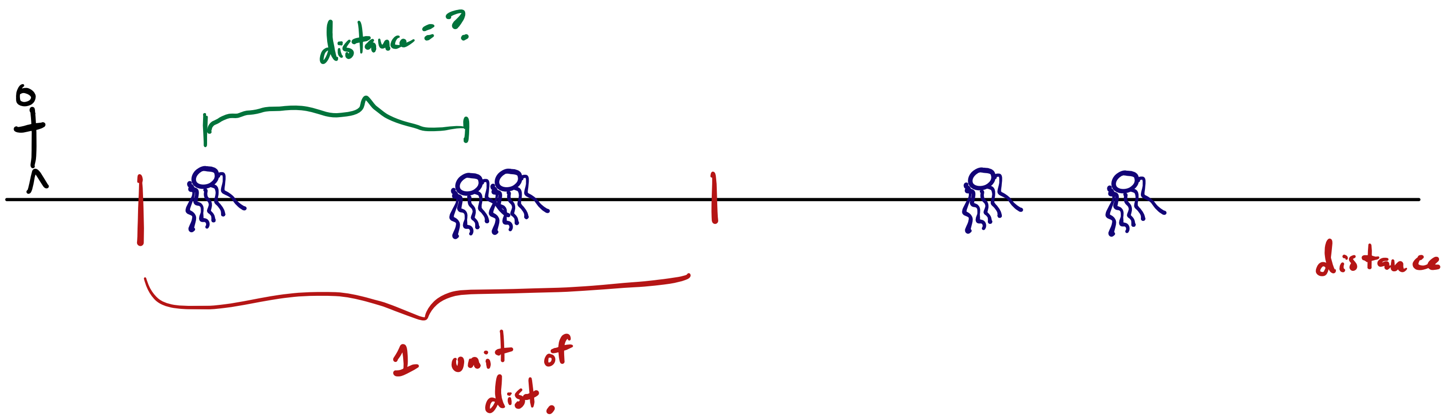
④ $P(X_7 \geq 150) = 1 - P(X_7 \leq 149)$

$= 1 - \sum_{x=0}^{149} \frac{e^{-140}\, 140^x}{x!}$

$= 1 - \text{ppois}(149, 140)$

↑ x ↑ λ

$= 0.2095.$



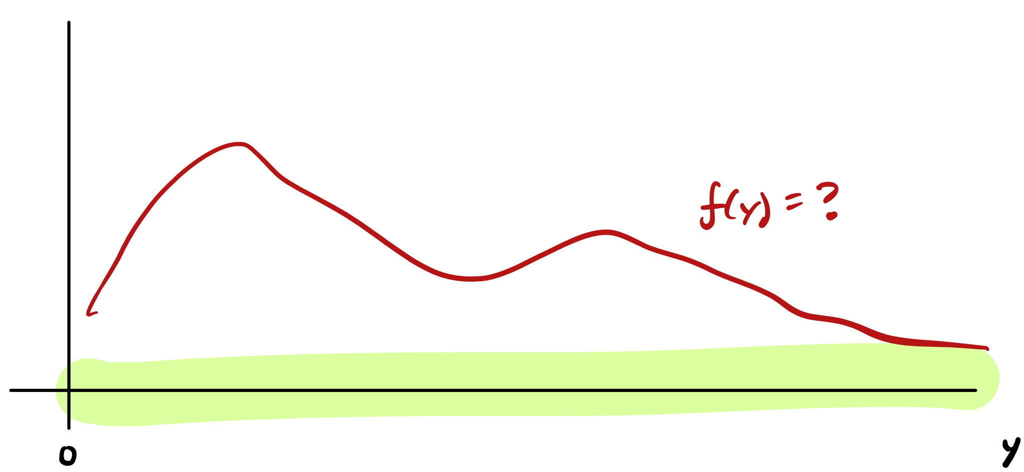
X = # in 1 unit of distance $\qquad \mathcal{X} = \{0, 1, 2, \dots\}$

PMF $\qquad p(x) = \dfrac{e^{-\lambda} \lambda^{x}}{x!}$

Y = distance from 1 to the next.

$\mathcal{Y} = [0, \infty)$

PDF $\qquad f(y)$



Find an expression for

$$P(Y > y) = P(\text{Most walk the distance } y \text{ before seeing a jelly fish})$$

↑ some value > 0.

$p(x) = \frac{e^{-\lambda} \lambda^x}{x!}$

$$= P(\text{I see 0 jelly fish in the distance } y)$$

Let X_y = # jellies I see in distance y.

$\sim \text{Poisson}(y\lambda)$

$$= P(X_y = 0)$$

$$= \frac{e^{-y\lambda}(y\lambda)^0}{0!}$$

$$= e^{-y\lambda}.$$

=>

$$P(Y \leq y) = 1 - P(Y > y)$$

$$= 1 - e^{-y\lambda}$$

Let $F(y) = 1 - e^{-y\lambda}$ for $y > 0$.

Then F is the cumulative dist. function (CDF) of Y.

The pdf $f(y)$ is

$$f(y) = \frac{d}{dy} F(y) = \lambda e^{-y\lambda} \quad \text{for } y > 0.$$

Consider the time between occurrences in a Poisson process...

Exponential distribution

The continuous probability distribution with pdf and cdf given by

$$f(y) = \lambda e^{-y\lambda}$$

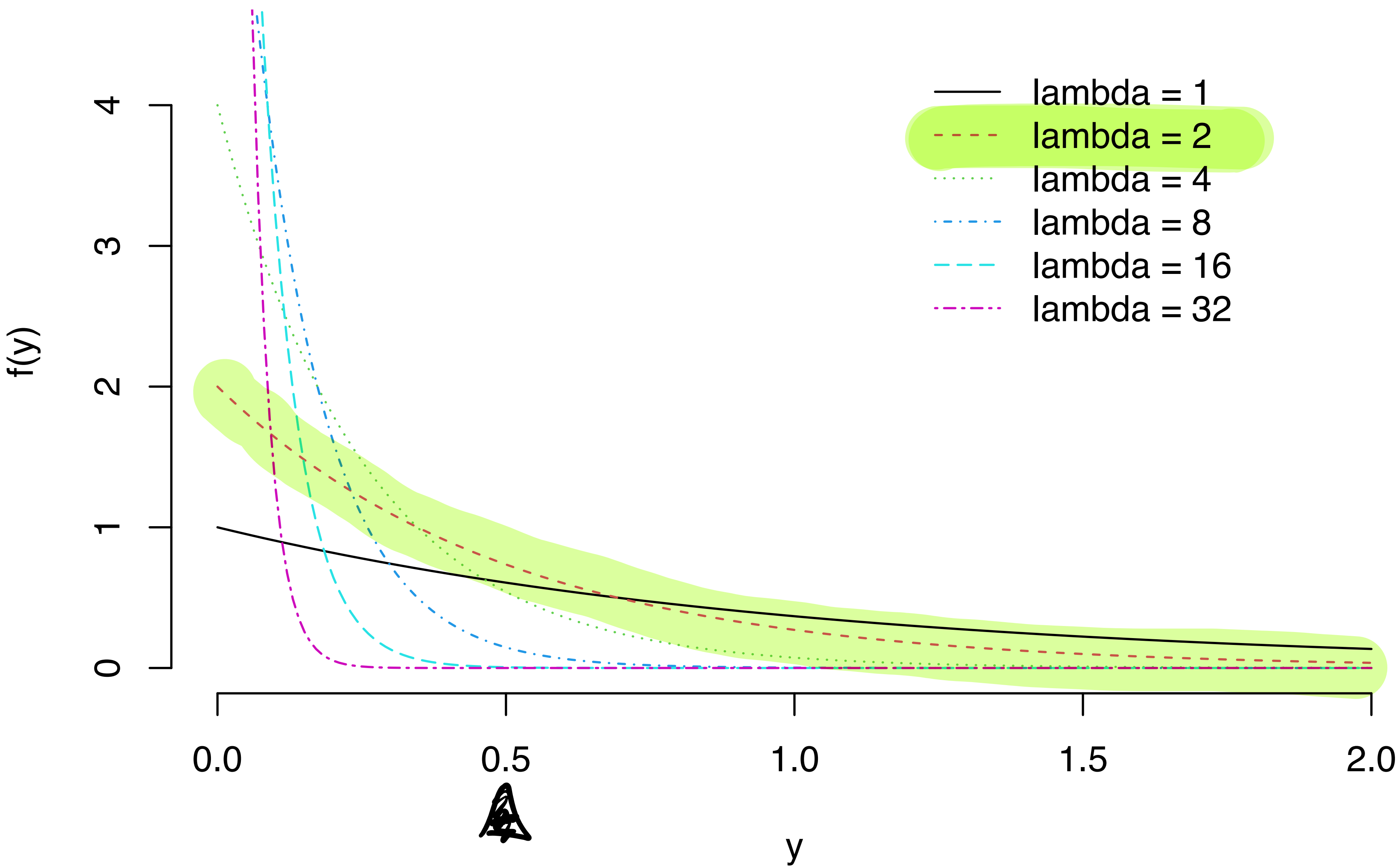
$$P(Y \leq y) \longrightarrow F(y) = 1 - e^{-y\lambda} \quad \text{for } y > 0$$

with $\lambda > 0$ is called the *Exponential distribution*.

We write $Y \sim \text{Exponential}(\lambda)$.

Derive: Start with $P(Y > y) = P(\text{no occurrences before time } y)$.

$f(y) = \lambda e^{-y\lambda}, \quad y > 0$



Mean and variance of Exponential distribution

If $Y \sim \text{Exponential}(\lambda)$ then

- $\mathbb{E}Y = 1/\lambda$.
- $\text{Var } Y = 1/\lambda^2$.

Exercise: Suppose the occurrence of car accidents in a town is a Poisson process with the expected number of accidents per day equal to 20.

1. What is the expected time between car accidents?
2. What is the probability that an accident happens in the next hour?

(1) $Y =$ time between. Then $Y \sim \text{Exponential}(\lambda = 20)$

$$\mathbb{E}Y = \frac{1}{20}$$

(2) $P\left(Y \leq \frac{1}{24}\right) = 1 - e^{-\left(\frac{1}{24}\right)20} = 1 - e^{-\frac{5}{6}} = 0.5654$

$F(y) = 1 - e^{-y\lambda}$

Exercise: Suppose the occurrence of blown-out tires along a freeway is a Poisson process with the expected number of blown-out tires per mile equal to $1/3$.

Find the probabilities of the following:

1. finding 2 blown-out tires in the next mile.
2. finding at least one blown-out tire in the next mile.
3. finding fewer than 3 blown-out tires in the next 12 miles.
4. finding a blown-out tire before going 5 miles.
5. not finding a blown-out tire in the next 3 miles.
6. finding a blown-out tire exactly 3 miles down the road.

$p(x) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0, 1, 2, \ldots$

$P(X \geq 1) = P(Y \leq 1)$

$P(Y = 3) = 0$

X = # tires per mile $\quad X \sim \text{Poisson}(\lambda = 1/3)$

① $P(X = 2) = \frac{e^{-1/3}(1/3)^2}{2!} = \text{dpois}(2, 1/3) = 0.0398$

② $P(X \geq 1) = 1 - P(X=0)$

$$= 1 - \frac{e^{-1/3}(1/3)^0}{0!}$$

$$= 1 - \text{dpois}(0, 1/3)$$

$$= 0.2835$$

❸ finding fewer than 3 blown-out tires in the next 12 miles.

③ $X_{12} \sim \text{Poisson}(\lambda = \underbrace{12 * \tfrac{1}{3}}_{4})$

$$P(X_{12} < 3) = P(X_{12} \leq 2)$$

$$= P(X_{12} = 0) + P(X_{12} = 1) + P(X_{12} = 2)$$

$$= \sum_{x=0}^{2} P(X_{12} = x)$$

$$= \sum_{x=0}^{2} \frac{e^{-4} 4^x}{x!}$$

On exam stop here

$$= \text{ppois}(\underset{x}{2}, \underset{\lambda}{4})$$

$$= 0.2381$$

4. finding a blown-out tire before going 5 miles.

Let Y = distance between tires. (Y is continuous)

$$P(Y < 5) = P(Y \le 5)$$
$$= F(5)$$
$$= 1 - e^{-(5)\frac{1}{3}}$$
$$= 1 - e^{-5/3}$$
$$= 0.8111$$

5. not finding a blown-out tire in the next 3 miles.

$$P(Y > 3) = 1 - P(Y \le 3)$$
$$= 1 - \left[1 - e^{-(3)\frac{1}{3}}\right]$$
$$= e^{-1}$$
$$= 0.3679$$