# STAT 515 Lec 12 slides

## Confidence interval for the mean when variance unknown

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These slides are an instructional aid; their sole purpose is to display, during the lecture, definitions, plots, results, etc. which take too much time to write by hand on the blackboard. They are not intended to explain or expound on any material.

**Recall**: If $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Normal}(\mu, \sigma^2)$, then

$$\bar{X}_n \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

is a $(1 - \alpha) \times 100\%$ CI for $\mu$.

*But what if we don't know $\sigma$?*

Using $\bar{X}_n \pm z_{\alpha/2} S_n / \sqrt{n}$ is okay if $n$ is large, but not if $n$ is small. . .

## Sampling distribution of "studentized" mean

If $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Normal}(\mu, \sigma^2)$, then $\dfrac{\bar{X}_n - \mu}{S_n/\sqrt{n}} \sim t_{n-1}$.

In the above, $t_{n-1}$ represents the $t$-distribution with $n-1$ degrees of freedom...

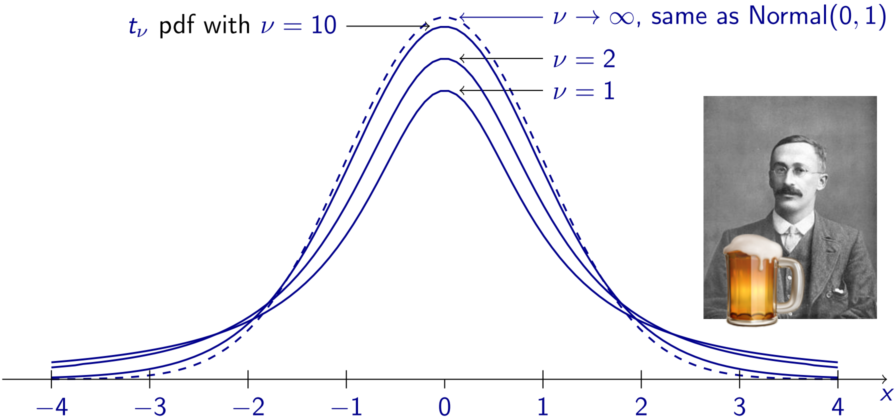
## The $t$-distributions

The probability distribution with pdf given by

$$f(x) = \frac{\Gamma(\frac{\nu+1}{2})}{\nu\pi\Gamma(\frac{\nu}{2})}\left(1+\frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}}, \quad \text{where } \Gamma(z) = \int_0^\infty u^{z-1}e^{-u}du,$$

for $\nu > 0$ is called the *t-distribution* with *degrees of freedom* $\nu$.

For a random variable $T$ with this distribution we write $T \sim t_\nu$.


$t_\nu$ pdf with $\nu = 10$
$\nu \to \infty$, same as Normal$(0, 1)$
$\nu = 2$
$\nu = 1$
$-4$
$-3$
$-2$
$-1$
$0$
$1$
$2$
$3$
$4$
$x$

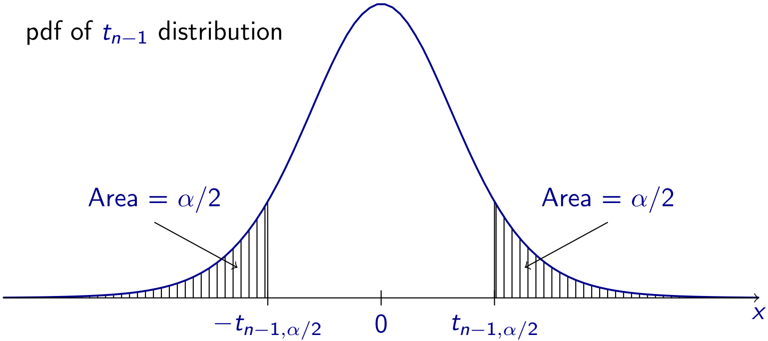
## How to build a $t_\nu$ random variable

If $Z \sim \text{Normal}(0,1)$ and $W \sim \chi^2_\nu$ are independent rvs, then

$$T = \frac{Z}{\sqrt{W/\nu}} \sim t_\nu.$$

**Discuss:** From the above we can show that $\dfrac{\bar{X}_n - \mu}{S_n/\sqrt{n}} \sim t_{n-1}$.

Upper quantile notation for the $t$ distributions:




Can use function `qt()` or a t-table to look up the values, e.g.

$$t_{19,0.025} = \texttt{qt(.975,19)} = 2.093024$$
$$t_{19,0.005} = \texttt{qt(.995,19)} = 2.860935.$$

## Confidence interval for mean of a Normal population with $\sigma$ unknown

Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Normal}(\mu, \sigma^2)$. Then a $(1-\alpha) \times 100\%$ CI for $\mu$ is

$$\bar{X}_n \pm t_{n-1,\alpha/2} \frac{S_n}{\sqrt{n}}.$$

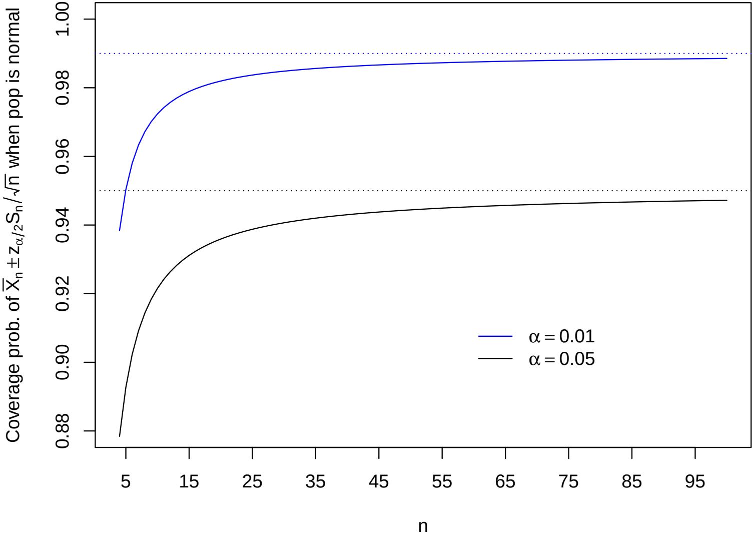
**Discuss:** How to obtain the above formula.

**Exercise:** These are the diameters of a sample of live oak acorns.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 0.455 | 0.375 | 0.428 | 0.390 | 0.388 | 0.420 | 0.430 | 0.420 | 0.350 |
| 0.440 | 0.450 | 0.430 | 0.500 | 0.480 | 0.460 | 0.429 | 0.427 | 0.424 |
| 0.512 | 0.428 | 0.426 | 0.430 | 0.400 | 0.460 | 0.440 | 0.400 | 0.450 |
| 0.410 | 0.398 | 0.420 | 0.453 | 0.449 | 0.428 | | | |

1. Make a Q-Q plot to check Normality of the population.
2. Construct a 95% confidence interval for the mean commute time of all students.
3. Construct a 99% confidence interval for the mean commute time of all students.
4. What if the intervals $\bar{X}_n \pm z_{\alpha/2} \cdot S_n/\sqrt{n}$ are used? How are they different?

```
X <- c(0.455,0.375,0.428,0.390,0.388,0.420,0.430,0.420,0.350,0.440,
       0.450,0.430,0.500,0.480,0.460,0.429,0.427,0.424,0.512,0.428,
       0.426,0.430,0.400,0.460,0.440,0.400,0.450,0.410,0.398,0.420,
       0.453,0.449,0.428)
qqnorm(X)
Xbar <- mean(X)
Sn <- sd(X)
n <- length(X)
alpha <- 0.05
tval <- qt(1-alpha/2,n-1)
lo <- Xbar - tval * Sn / sqrt(n)
up <- Xbar + tval * Sn / sqrt(n)
```

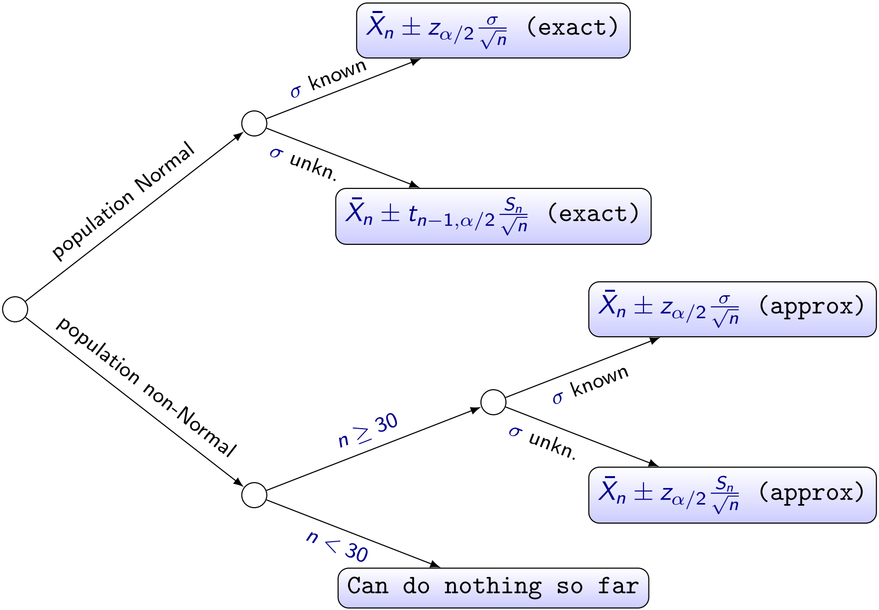

Coverage prob. of $\overline{X}_n \pm z_{\alpha/2} S_n/\sqrt{n}$ when pop is normal
1.00
0.98
0.96
0.94
0.92
0.90
0.88
5
15
25
35
45
55
65
75
85
95
n
$\alpha = 0.01$
$\alpha = 0.05$

## CI for mean of non-Normal population with $\sigma$ unknown

Let $X_1, \dots, X_n$ be a rs from a pop. with mean $\mu$ and variance $\sigma^2 < \infty$, then

$$\bar{X}_n \pm z_{\alpha/2} \cdot \frac{S_n}{\sqrt{n}}$$

is an approximate $(1-\alpha) \times 100\%$ CI for $\mu$ when $n$ is large ($\geq 30$, say).


population Normal
$\sigma$ known
$\bar{X}_n \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$ (exact)
$\sigma$ unkn.
$\bar{X}_n \pm t_{n-1,\alpha/2}\frac{S_n}{\sqrt{n}}$ (exact)
population non-Normal
$n \geq 30$
$\sigma$ known
$\bar{X}_n \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$ (approx)
$\sigma$ unkn.
$\bar{X}_n \pm z_{\alpha/2}\frac{S_n}{\sqrt{n}}$ (approx)
$n < 30$
Can do nothing so far