STAT 515 Lec 14 slides

Hypothesis testing

Karl Gregory

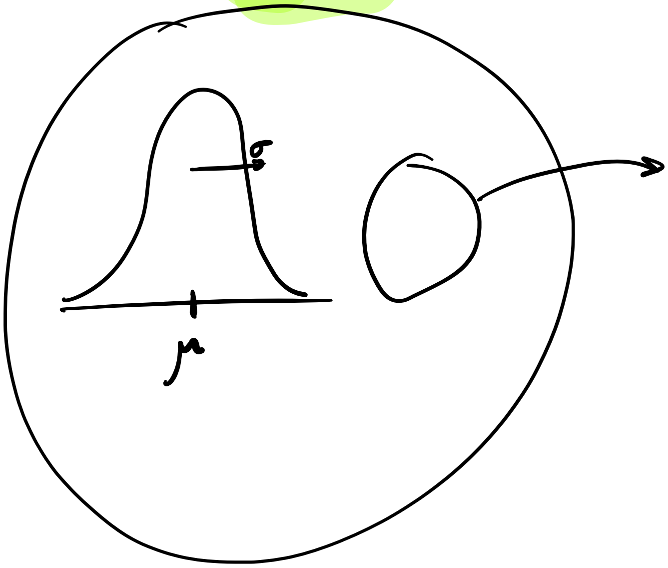
University of South Carolina

These slides are an instructional aid; their sole purpose is to display, during the lecture, definitions, plots, results, etc. which take too much time to write by hand on the blackboard. They are not intended to explain or expound on any material.

Want to learn about these.

pop μ unknown

σ^2 (known)



$X_1, \dots, X_n$

$\bar{X}_n$

S_n^2

Statistical Inference $\longrightarrow$ Prediction / forecasting

Confidence Interval

Testing hypotheses

Testing hypotheses about μ based $X_1, \dots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

Hypothesis is some statement about μ.

Consider two contradicting hypotheses

(i) null hypothesis H_0 "H nought"

(ii) alternate hypothesis H_1

Based on data, we will either

(i) Reject H_0

(ii) Fail to reject H_0

A *statistical inference* is a conclusion about a pop. parameter based on a rs.

Specifically, a decision concerning contradictory statements about the parameter:

- The *null hypothesis* H_0.
- The *alternate hypothesis* H_1.

The decision is whether to

1. Reject H_0, thereby concluding that H_1 is true.
2. Not reject H_0, thereby not concluding anything.

A *test of hypotheses* is a rule for when to reject H_0 based on the data.

$p = ?$

Exercise: We want to know whether a coin is unfair. Let p be the prob. of heads.

We want to test $\boxed{H_0\colon p = 1/2}$ versus $H_1\colon p \neq 1/2$ based on $n = 100$ coin tosses.

Discuss the following:

1. Reject or fail to reject H_0 if 51 heads observed?
2. Reject or fail to reject H_0 if 60 heads observed?
3. Reject or fail to reject H_0 if 90 heads observed?
4. Reject or fail to reject H_0 if 50 heads observed?
5. What possible evidence could convince us that $p = 1/2$?
6. If the coin is fair, find prob. of observing a # of heads ≥ 60 or ≤ 40.

Statistical inference with hypothesis testing

$H_0: p = \frac{1}{2}$

$H_1: p \neq \frac{1}{2}$

Truth

Inference	H_0 true	H_0 false
Reject H_0	Incorrect Type I error	correct
Fail to reject H_0	correct	Incorrect Type II error

Calibrate our rule for rejecting H_0 by ensuring $P(\text{Type I error}) \leq \alpha$, typically $\alpha = 0.05$

"significance level".

Exercise: Is a treatment effective in lowering cholesterol levels? Let μ represent the average difference (after-minus-before treatment) in cholesterol levels.

We want to test H_0: $\mu \geq 0$ versus H_1: $\mu < 0$ with data from $n = 100$ subjects.

Discuss the following:

1. Suppose we obtain $\bar{X}_n = 10.0$. Should we reject H_0?
2. What if we had observed $\bar{X}_n$ equal to -10.0?
3. If the changes in chol. level are $\mathcal{N}(0, (25)^2)$, find $P(\bar{X}_n < -10.0)$.

Our data may lead us to an incorrect decision about H_0 and H_1:

- A *Type I error* is rejecting H_0 when H_0 is true.
- A *Type II error* is failing to reject H_0 when H_0 is false.

We like to calibrate our tests of hypotheses such that $P(\text{Type I error}) \leq \alpha$.

Then we call α the *significance level* of the test.

Discuss: Table summarizing possible outcomes of inference.

We want to test hypotheses like

"null value"

1. $H_0: \mu \le \mu_0$ vs $H_1: \mu > \mu_0$ Right-sided
2. $H_0: \mu \ge \mu_0$ vs $H_1: \mu < \mu_0$ Left-sided
3. $H_0: \mu = \mu_0$ vs $H_1: \mu \ne \mu_0$ Two-sided

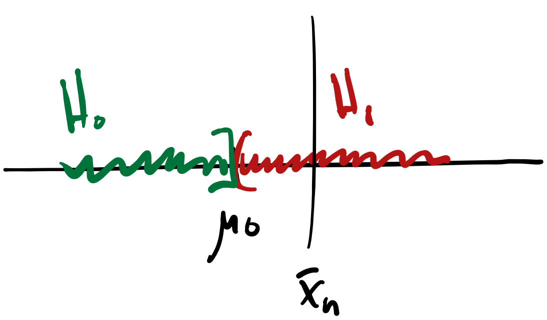
One-sided

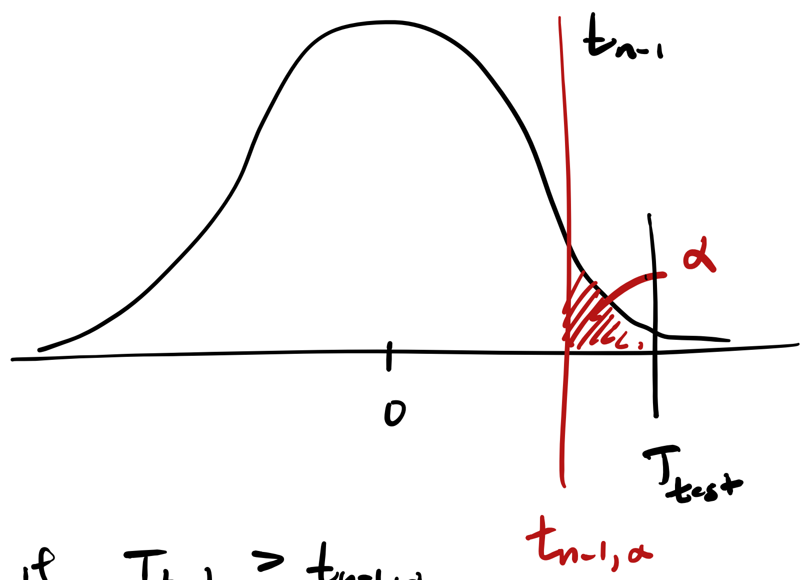
How to decide whether to reject H_0?

Compute the test statistic

$$T_{test} = \frac{\bar{X}_n - \mu_0}{S_n/\sqrt{n}}$$

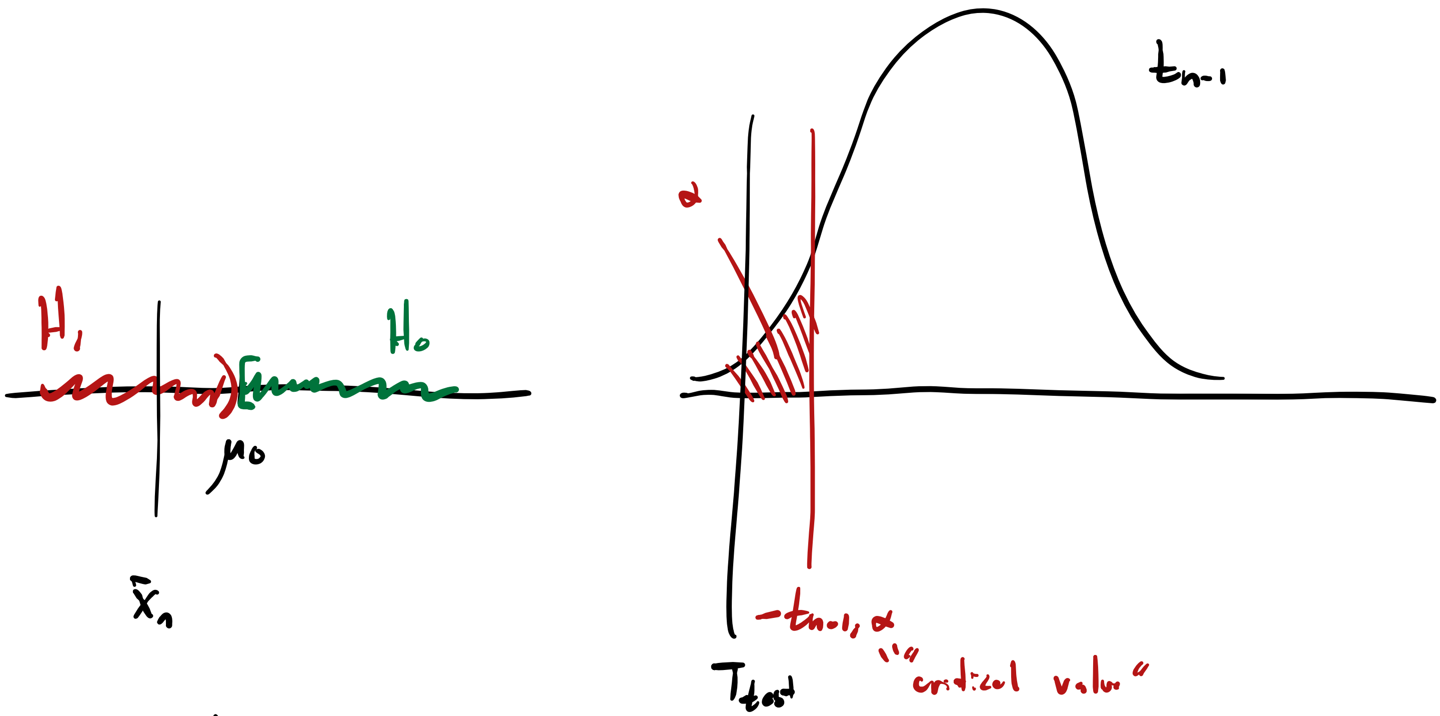
1. $H_0: \mu \le \mu_0$ vs $H_1: \mu > \mu_0$





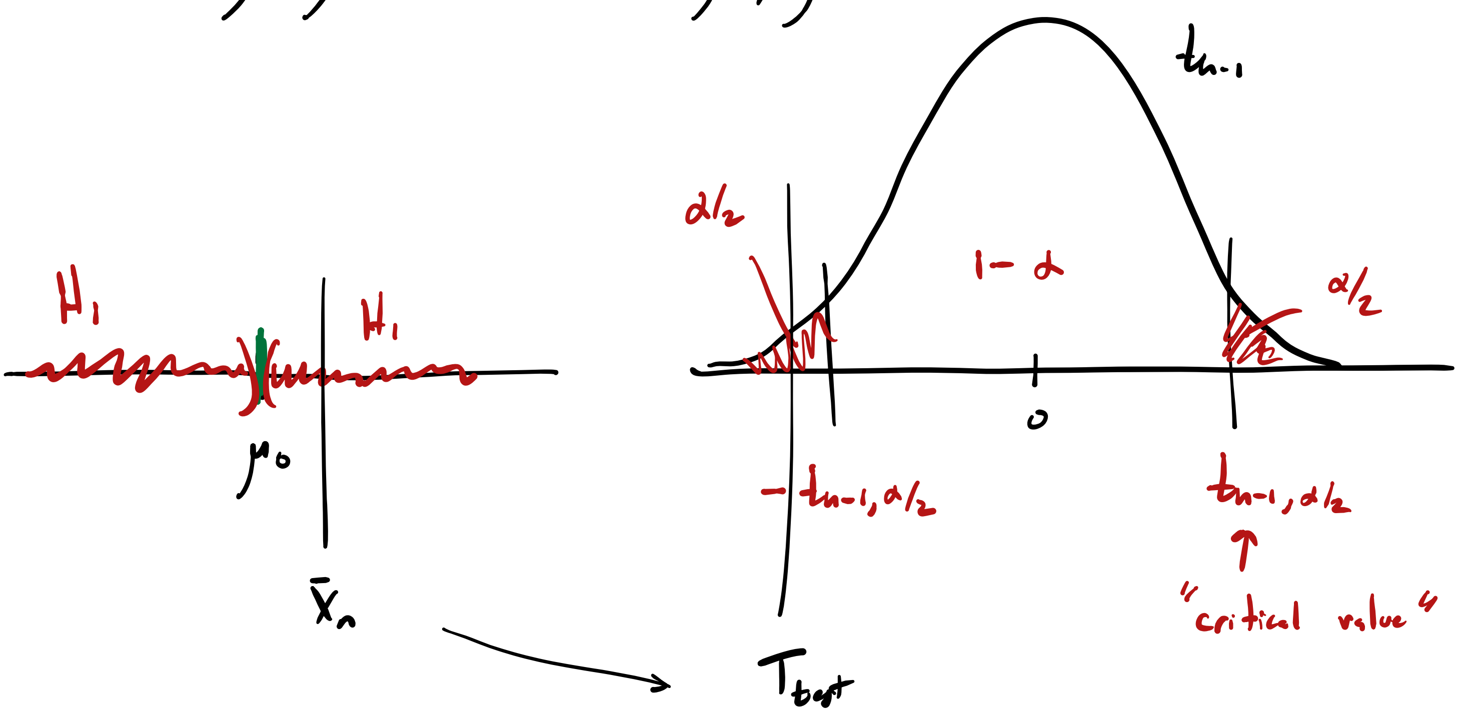
Reject H_0 if $T_{test} > t_{n-1,\alpha}$

2. $H_0: \mu \geq 0$ vs $H_1: \mu < \mu_0$



Reject H_0 if $T_{test} < -t_{n-1,\alpha}$

3. $H_0: \mu = \mu_0$ vs $H_1: \mu \neq \mu_0$



Reject H_0 if $T_{test} < -t_{n-1,\alpha/2}$ or $T_{test} > t_{n-1,\alpha/2}$

Reject H_0 if $|T_{test}| > t_{n-1,\alpha/2}$.

Suppose $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Normal}(\mu, \sigma^2)$, with μ and σ^2 unknown.

We will consider null and alternate hypotheses of the form

$$\begin{array}{lcl} H_0\colon \mu \geq \mu_0 & \text{or} & H_0\colon \mu = \mu_0 & \text{or} & H_0\colon \mu \leq \mu_0 \\ H_1\colon \mu < \mu_0 & & H_1\colon \mu \neq \mu_0 & & H_1\colon \mu > \mu_0. \end{array}$$

Here μ_0 is a value specified by the researcher called the *null value*.

Exercise: For each set of hypotheses, find a test based on the *test statistic*

$$T_{\text{test}} = \frac{\bar{X}_n - \mu_0}{S_n/\sqrt{n}}$$

with $P(\text{Type I error}) \leq \alpha$.

Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Normal}(\mu, \sigma^2)$, with μ and σ^2 unknown.

Tests about μ when σ is unknown

For some null value μ_0, define the test statistic

$$T_{\text{test}} = \frac{\bar{X} - \mu_0}{S_n/\sqrt{n}}.$$

Then the following tests have $P(\text{Type I error}) \leq \alpha$.

$H_0\colon \mu \geq \mu_0$ $H_1\colon \mu < \mu_0$	$H_0\colon \mu = \mu_0$ $H_1\colon \mu \neq \mu_0$	$H_0\colon \mu \leq \mu_0$ $H_1\colon \mu > \mu_0$
Reject H_0 if $T_{\text{test}} < -t_{n-1,\alpha}$	Reject H_0 if $\lvert T_{\text{test}} \rvert > t_{n-1,\alpha/2}$	Reject H_0 if $T_{\text{test}} > t_{n-1,\alpha}$

Exercise: Suppose a bottler of soft-drinks claims that its bottling process results in an internal pressure of 157 psi. You want to know whether the mean pressure is less than 157 (Ex 6.92 in [1]).

1. What are the relevant hypotheses?
2. Based on a sample of size $n = 30$ you get $\bar{X} = 155.7$ and $S_n = 3.0$. What is your inference at the $\alpha = 0.05$ significance level?
3. Identify the following as a correct decision, a Type I error, or a Type II error:
   a. Suppose $\mu = 157.5$ and your data leads you to reject H_0.
   b. Suppose $\mu = 157.5$ and your data leads you to not reject H_0.
   c. Suppose $\mu = 156.5$ and your data leads you to reject H_0.
   d. Suppose $\mu = 156.5$ and your data leads you to not reject H_0.

Bottles of soft drinks bottles supposedly at 157 psi:

Is the mean pressure actually less than 157 psi?

put in alternate hypothesis

Suppose X is psi of a randomly selected bottle,

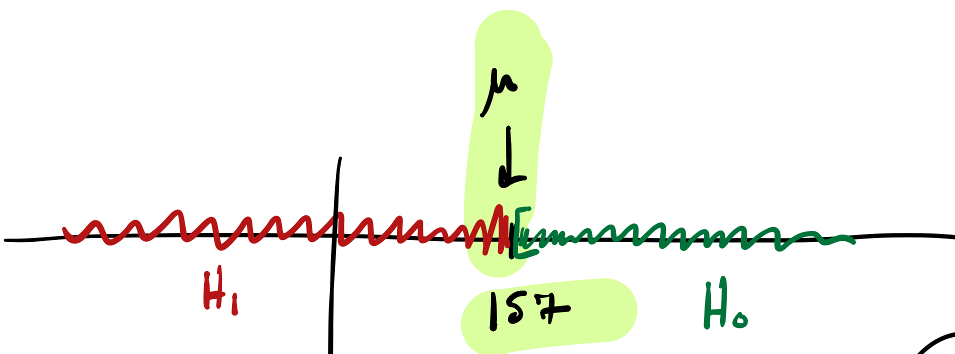
$$X \sim N(\mu, \sigma^2)$$

μ_0, null value

$H_0: \mu \geq 157$

$H_1: \mu < 157$

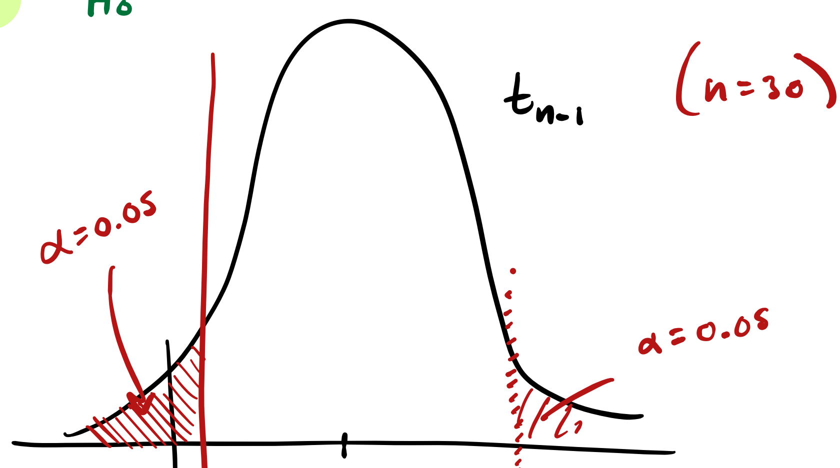
I observe $\bar{x}_n = 155.7$, $S_n = 3.0$, $n = 30$.



$\bar{x}_n = 155.7$

$$T = \frac{\bar{X}_n - 157}{S_n/\sqrt{n}}$$

$$\frac{155.7 - 157}{3.0/\sqrt{30}} = -2.373$$



$t_{n-1,0.05} = 1.6991$

$-1.6991 = -t_{n-1,0.05}$

I should reject H_0 if

$$\frac{\bar{x}_n - 157}{S_n/\sqrt{n}} < -1.6991.$$

Exercise: The average height of 14 randomly selected ten-yr-old Loblolly pine trees was $\bar{X}_n = 27.44$ and the sample standard deviation was $S_n = 1.54$. Assume that the heights of ten-yr-old Loblolly pine trees are Normally distributed.

1. Test the hypotheses H_0: $\mu \leq 26$ versus H_1: $\mu > 26$ at $\boxed{\alpha = 0.05.}$
2. Test the hypotheses H_0: $\mu \geq 26$ versus H_1: $\mu < 26$ at $\alpha = 0.05$.
3. Test the hypotheses H_0: $\mu = 26$ versus H_1: $\mu \neq 26$ at $\alpha = 0.05$.
4. Build a 95% CI for μ.

1

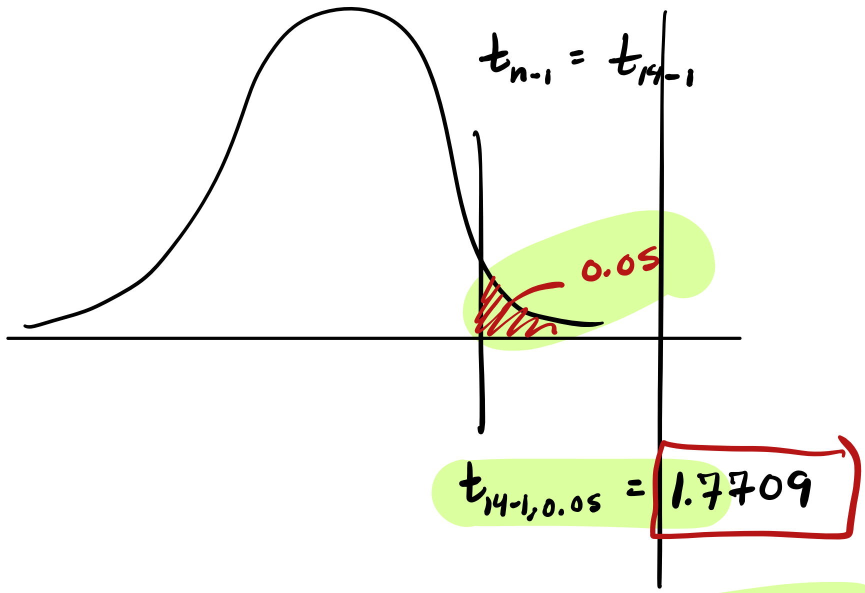
$n = 14$ $\mu_0 = 26$

$\bar{X}_n = 27.44$

$S_n = 1.54$

$$T_{test} = \frac{\bar{X}_n - \mu_0}{S_n/\sqrt{n}} = \frac{27.44 - 26}{1.54/\sqrt{14}} = 3.499$$

$H_0: \mu \leq 26$ vs $H_1: \mu > 26$

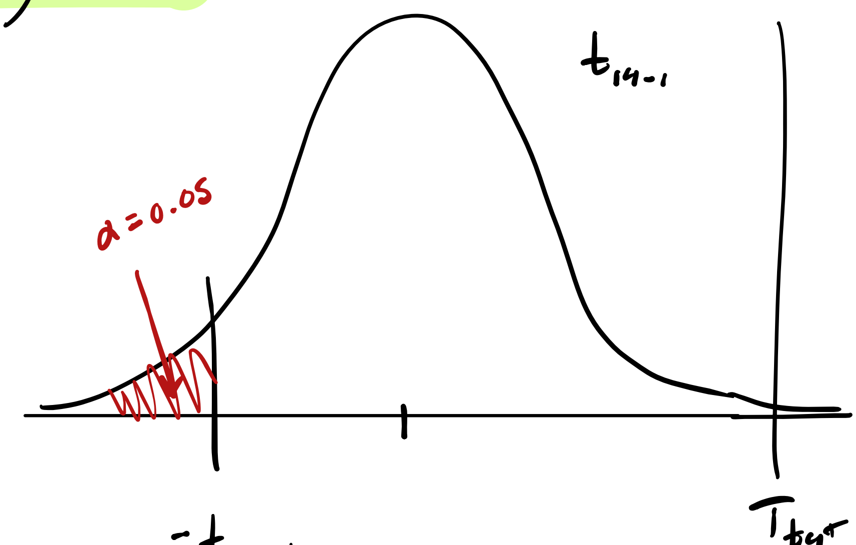


$T_{test} = 3.499$

② $H_0: \mu \geq 26$ vs $H_1: \mu < 26$

$\bar{x}_n = 27.44$

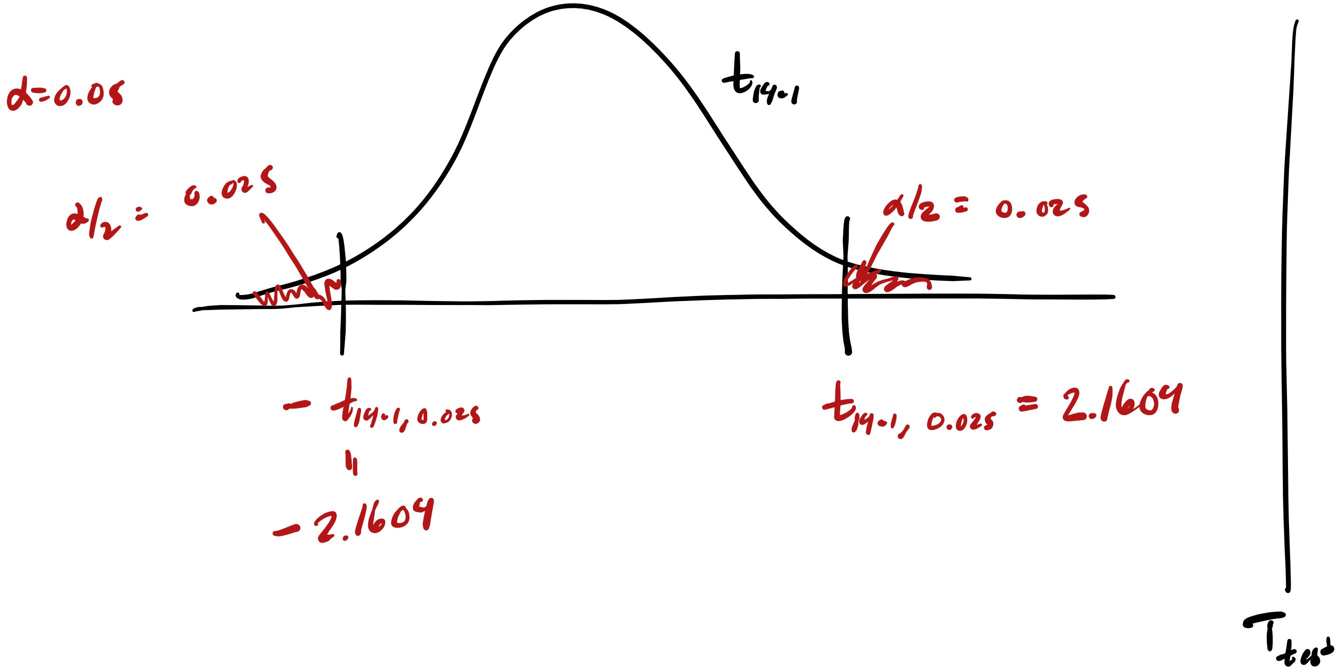
$T_{test} = 3.499$



$-t_{n-1,\alpha}$
$\parallel$
$-t_{14-1,0.05} = -1.7709$

③ $H_0: \mu = 26$ vs $H_1: \mu \neq 26$

$T_{test} = 3.499$



<u>Recall:</u> A $(1-\alpha)*100\%$ C.I. for μ is

$$\bar{X}_n \pm t_{n-1,\alpha/2} \frac{S_n}{\sqrt{n}} .$$

$$\mu_0 \in \left[\bar{X}_n - t_{n-1,\alpha/2} \frac{S_n}{\sqrt{n}}, \ \bar{X}_n + t_{n-1,\alpha/2} \frac{S_n}{\sqrt{n}} \right]$$

$$\Longleftrightarrow \quad |T_{test}| < t_{n-1,\alpha/2}$$

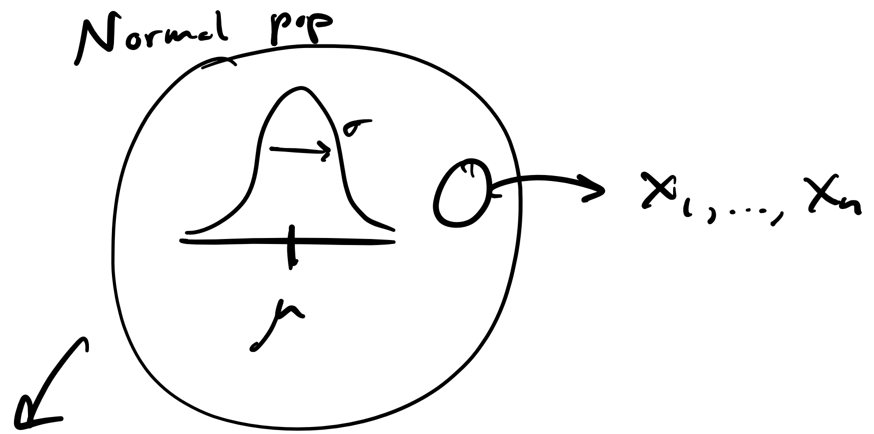
That is, we fail to reject $H_0: \mu = \mu_0$ at significance level α.

$$T_{test} = \frac{\bar{X}_n - \mu_0}{S_n/\sqrt{n}}$$

For two-sided tests at α, just see if $(1-\alpha)100\%$ CI contains the null value!

For H_0: $\mu = \mu_0$ versus H_1: $\mu \neq \mu_0$ we have:

$$|T_{\text{test}}| \leq t_{n-1,\alpha/2} \iff \mu_0 \in \left[\bar{X}_n - t_{n-1,\alpha/2}\frac{S_n}{\sqrt{n}}, \bar{X}_n + t_{n-1,\alpha/2}\frac{S_n}{\sqrt{n}}\right].$$



1. Introduction to hypothesis testing
2. Testing hypotheses about μ under Normality
3. **Testing hypotheses about μ when data is not Normal**
4. Testing hypotheses about p

Since $\sqrt{n}(\bar{X}_n - \mu)/S_n$ behaves like $Z \sim \mathcal{N}(0,1)$ for large n...

Tests about μ when data non-Normal and $\boxed{n \geq 30}$

For some null value μ_0, define the test statistic

$$T_{\text{test}} = \frac{\bar{X} - \mu_0}{S_n/\sqrt{n}}.$$

Then the following tests have $P(\text{Type I error}) \leq \alpha$.

$H_0\colon \mu \geq \mu_0$ $H_1\colon \mu < \mu_0$	$H_0\colon \mu = \mu_0$ $H_1\colon \mu \neq \mu_0$	$H_0\colon \mu \leq \mu_0$ $H_1\colon \mu > \mu_0$
Reject H_0 if $T_{\text{test}} < -z_\alpha$	Reject H_0 if $\lvert T_{\text{test}}\rvert > z_{\alpha/2}$	Reject H_0 if $T_{\text{test}} > z_\alpha$

very close to t-dist. quantiles for large n.

Time allowing:

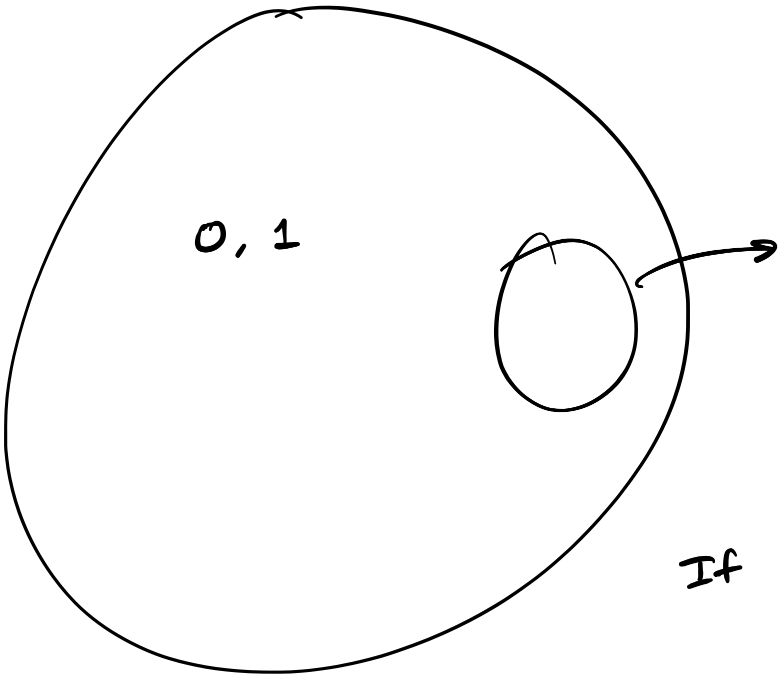
1. Draw a random sample of size $n = 35$ from the 2009 Boston Marathon `women's finishing times` and test the hypotheses

$$H_0\colon \mu \leq 4 \text{ versus } H_1\colon \mu > 4$$

at the $\alpha = 0.05$ significance level.

2. Repeat this 1000 times and record the proportion of times you reject H_0.

proportion of "successes" p, unknown



$X_1, \ldots, X_n \overset{ind}{\sim} \text{Bernoulli}(p)$.

$$\hat{p}_n = \bar{X}_n = \frac{1}{n}(X_1 + \cdots + X_n)$$

If n is large

$$Z = \frac{\hat{p}_n - p}{\sqrt{\frac{p(1-p)}{n}}} \overset{approx}{\sim} N(0,1).$$

Now we want to test

1. $H_0: p \le p_0$ vs $H_1: p > p_0$ "right-sided"
2. $H_0: p \ge p_0$ vs $H_1: p < p_0$ "left-sided"
3. H_0 $p = p_0$ vs $H_1: p \ne p_0$ "two-sided"

where p_0 is some "null value".

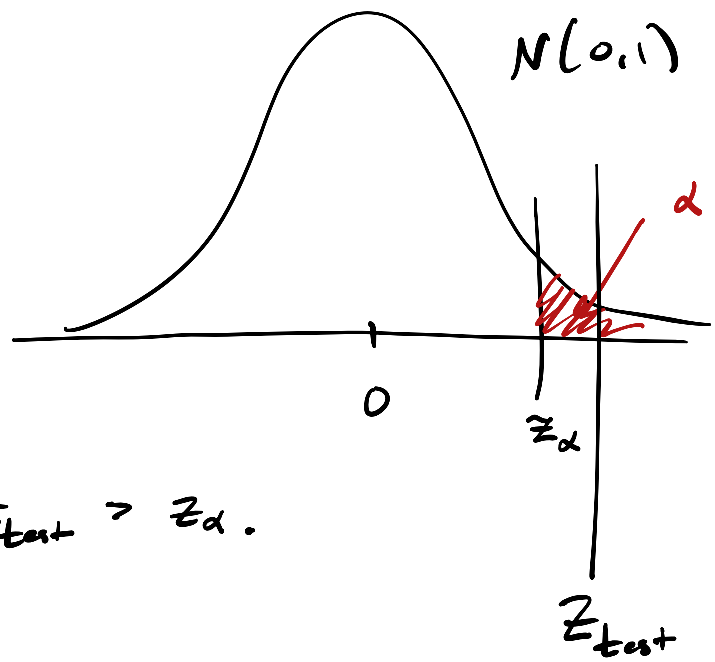
Define a test statistic

$$Z_{test} = \frac{\hat{p}_n - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} \overset{approx}{\underset{\text{if } p = p_0}{\sim}} N(0,1).$$

$$T_{test} = \frac{\bar{x}_n - \mu_0}{S_n / \sqrt{n}}$$

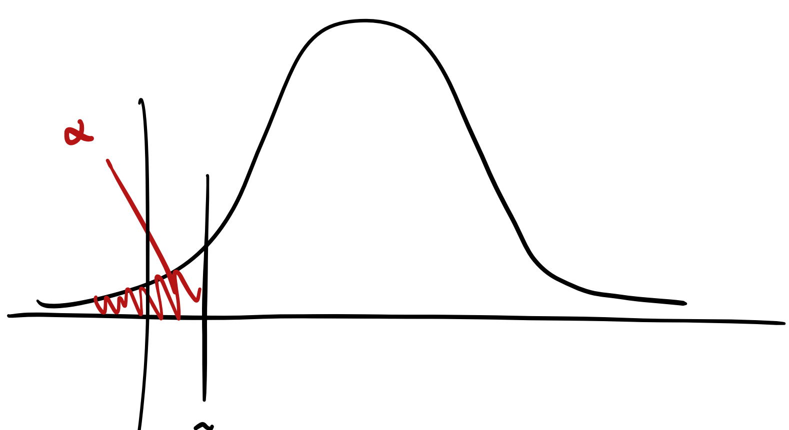
1. $H_0: p \le p_0$ vs $H_1: p > p_0$

$$Z_{test} = \frac{\hat{p}_n - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$$



Reject H_0 if $Z_{test} > z_\alpha$.

2. $H_0: p \ge p_0$ vs $H_1: p < p_0$

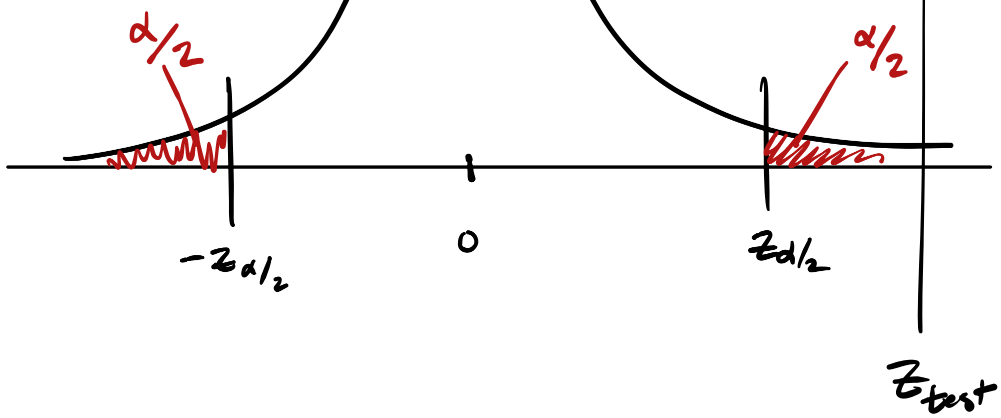


Reject H_0 if $Z_{test} < -z_\alpha$.

Z_{test}

3. $H_0: p = p_0$ vs $H_1: p \neq p_0$

$$Z_{test} = \frac{\hat{p}_n - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$$



Reject H_0 if $|Z_{test}| > z_{\alpha/2}$.

Since $\sqrt{n}(\hat{p}_n - p)/\sqrt{p(1-p)}$ behaves like $Z \sim \mathcal{N}(0,1)$ for large n...

Tests about p (for $np_0 \geq 15$ and $n(1-p_0) \geq 15$)

For some null value μ_0, define the test statistic

$$Z_{\text{test}} = \frac{\hat{p}_n - p_0}{\sqrt{\dfrac{p_0(1-p_0)}{n}}}.$$

Then the following tests have $P(\text{Type I error}) \leq \alpha$.

$H_0\colon p \geq p_0$ $H_1\colon p < p_0$	$H_0\colon p = p_0$ $H_1\colon p \neq p_0$	$H_0\colon p \leq p_0$ $H_1\colon p > p_0$
Reject H_0 if $Z_{\text{test}} < -z_\alpha$	Reject H_0 if $\lvert Z_{\text{test}} \rvert > z_{\alpha/2}$	Reject H_0 if $Z_{\text{test}} > z_\alpha$

1. $H_0: p \leq 1/2$ versus $H_1: p > 1/2$ $p =$ proportion of females.

$\hat{p}_n = \frac{287}{500} = 0.574$

$287 \times 2 = .574$

Exercise: Does a female-inhabiting parasite tip the sex ratio of its hosts' offspring in favor of females? A sample of size $\boxed{n = 500}$ offspring from parasite-infected females is collected, among which there are $\boxed{287}$ females.

1. What are the relevant hypotheses?
2. Carry out a test of the hypotheses at the $\alpha = 0.05$ significance level.
3. Identify the following as a correct decision, a Type I error, or a Type II error:
   a. Suppose $p = 0.60$ and your data leads you to reject H_0. correct
   b. Suppose $p = 0.60$ and your data leads you to not reject H_0. Type II
   c. Suppose $p = 0.50$ and your data leads you to reject H_0. Type I
   d. Suppose $p = 0.50$ and your data leads you to not reject H_0. correct

(2)

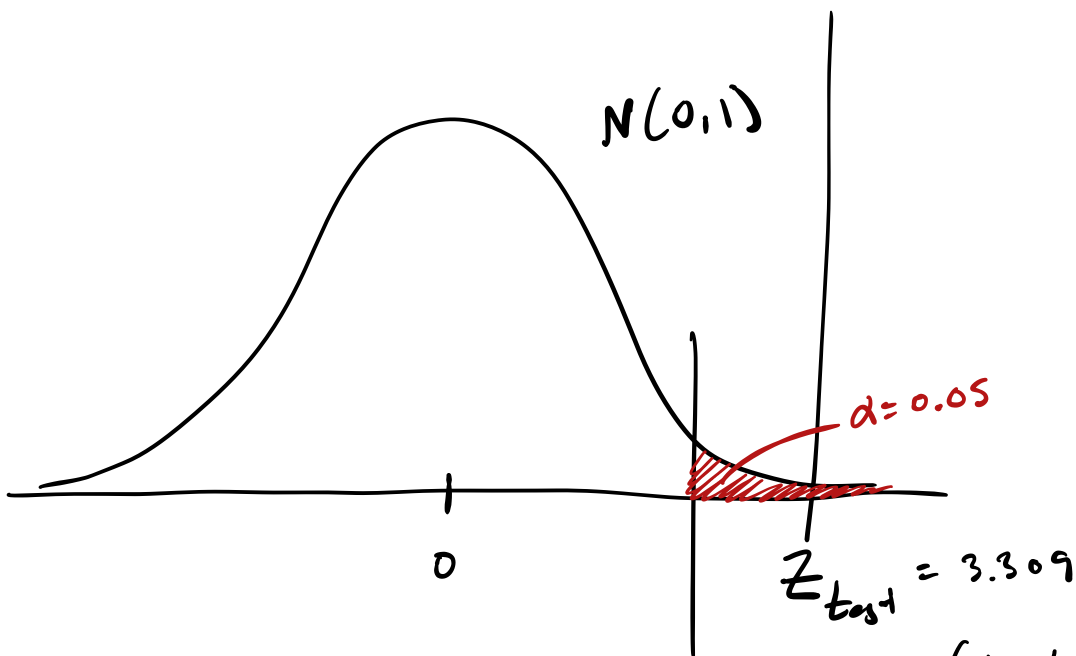
$\alpha = 0.05$ $\quad p_0 = 1/2$ $\quad \hat{p}_n = 0.574$

$n = 500$

$H_0: p \le 1/2$ vs $H_1: p > 1/2$

$$Z_{test} = \frac{0.574 - 0.5}{\sqrt{\frac{0.5(1-0.5)}{500}}} \qquad Z_{test} = \frac{\hat{p}_n - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$$

$$= 3.309$$



$z_{0.05} = 1.645$ (bottom row of t-table)

Exercise: In a tasting experiment, each of 121 blindfolded students was fed either a red or green gummy bear, (each with probability 1/2) and asked to identify the color from the taste. Of the $\boxed{121,\ 97}$ correctly identified the color(Ex. 8.82 of [1]).

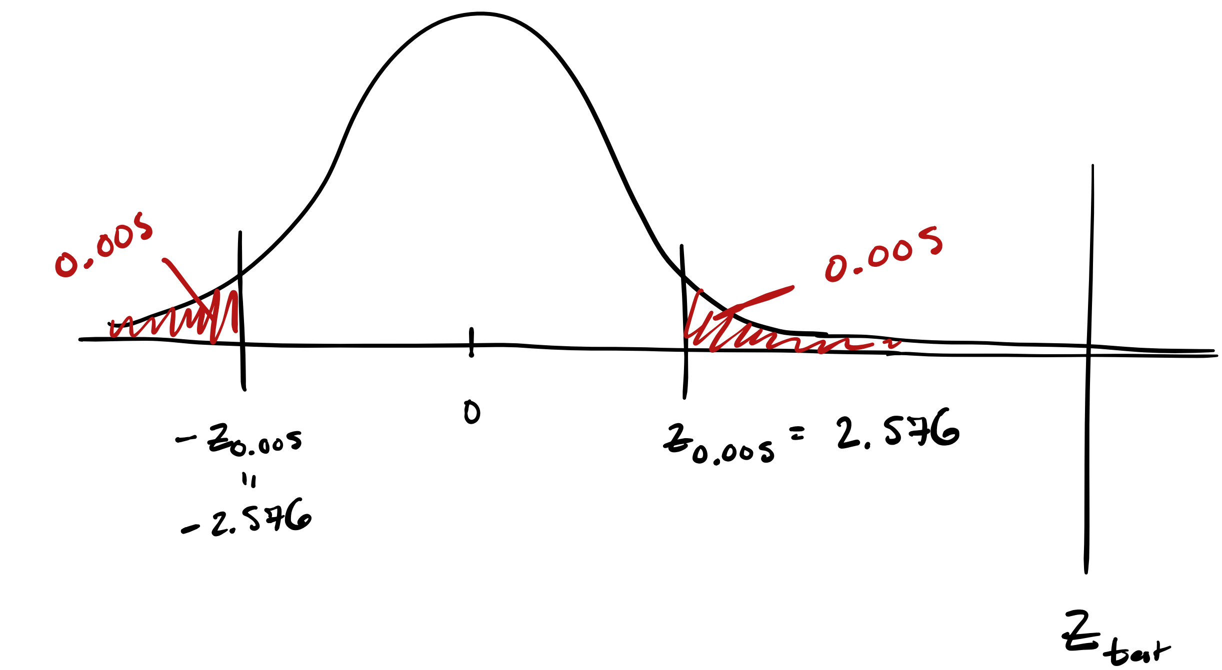
1. If the students guessed "red" or "green" based on flipping a coin, with what probability would they guess the color correctly? 1/2
2. Suppose you wish to know if the students are doing better or worse than guessing. What are the relevant hypotheses?
3. Test the hypotheses at the $\alpha = 0.01$ significance level.

2 $H_0: p = 1/2$ vs $H_1: p \neq 1/2$

3 $\alpha = 0.01$ $n = 121$ $\hat{p}_n = \frac{97}{121} = 0.802$

$$z_{test} = \frac{0.802 - 0.5}{\sqrt{\frac{0.5(1-0.5)}{121}}} = 6.644$$

$\alpha = 0.01$



Reject.

J.T. McClave and T.T. Sincich.
Statistics.
Pearson Education, 2016.