STAT 515 sp 2026 Exam II

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- Do not open this exam until told to do so.
- You may have two handwritten sheets of notes out during the exam.
- You have 75 minutes to work on this exam.
- You may NOT use any kind of calculator.

$X \sim$		$\mathcal{X}$	$\mathbb{E}X$	$\mathrm{Var}(X)$
Poisson(λ)	$P(X = x) = \dfrac{e^{-\lambda}\lambda^x}{x!}$	$x = 0, 1, 2, \ldots$	λ	λ
Exponential(λ)	$P(X \leq x) = 1 - e^{-x\lambda}$	$x > 0$	$\dfrac{1}{\lambda}$	$\dfrac{1}{\lambda^2}$

$$\hat{p}_n \pm z_{\alpha/2} \cdot \sqrt{\hat{p}_n(1 - \hat{p}_n)/n}$$

$$Z_{\text{test}} = \frac{\hat{p}_n - p_0}{\sqrt{p_0(1 - p_0)/n}}$$

$$\bar{X}_n \pm t_{n-1,\alpha/2} \cdot S_n/\sqrt{n}$$

$$T_{\text{test}} = \frac{\bar{X}_n - \mu_0}{S_n/\sqrt{n}}$$

A z-table and a t-table are appended to this exam.

$$P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

1. A type of mushroom pops up in a hermit's forest such that when he walks through his forest he encounters, on average, 5 mushrooms every 100 yards. Suppose the mushrooms pop up in the forest according to a Poisson process. Here comes the hermit now...

(a) Give the probability that he will walk 100 yards without encountering a single mushroom.

$$P(X=0) = \frac{e^{-5}5^0}{0!} = e^{-5}(1) = \boxed{e^{-5}}$$

(b) Give the probability that he encounters at least one mushroom in the first 100 yards.

$$P(X\geq 1) = 1 - P(X<1) = 1 - P(X=0)$$
$$\boxed{= 1 - e^{-5}}$$

(c) Give an expression for the probability that he finds at least 5 mushrooms in the first 100 yards. You do not need to evaluate your expression.

$$P(X\geq 5) = 1 - P(X\leq 4)$$
$$= 1 - \sum_{x=0}^{4} \frac{e^{-5}5^x}{x!}$$

(d) Give the expected number of mushrooms he will encounter if he walks 300 yards in his forest.

X_t = # occurence in time/space

$X_3 \sim \text{Poisson}(3\cdot 5)$ $\underset{15}{\shortparallel}$

$\boxed{EX = 15}$

$\frac{100}{300} = \frac{1}{3}$

(e) Suppose he measures the distance from the first mushroom he encounters to the next mushroom he encounters. What probability distribution does this random variable have?

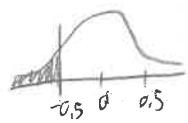
This random variable has a continuous probability distribution (exponential because it's measuring between events).

2. An espresso machine yields double shots with volumes following a normal distribution with mean 60 ml and standard deviation 2 ml.

(a) Give the probability that the next double shot will have volume less than 59 ml.

$X \sim Normal(60, 2^2)$, $\mu = 60$, $\sigma = 2$

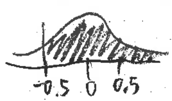
$P(X < 59) = P(Z < \frac{59-60}{2}) = P(Z < -0.5) = 0.5 - 0.1915 = \boxed{0.3085}$



(b) Give the probability that the next double shot will have volume greater than 59 ml.

$P(X > 59) = 1 - P(X \leq 59) = 1 - P(X < 59) = 1 - 0.3085 = \boxed{0.6915}$

$P(X > 59) = P(Z > -0.5) = 0.1915 + 0.50 = 0.6915$



(c) Give the 93.32th percentile of the distribution of volumes of double shots of espresso from this machine.

$P(Z < 1.50) = 0.9332$

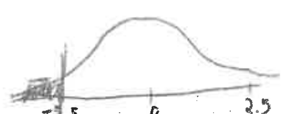
$\frac{X-60}{2} = 1.50$

$X = \boxed{63 \text{ mL}}$

(d) Suppose you pull 25 double shots of espresso. Give the probability that the average of the volumes of the 25 double shots is less than 59 ml.

$\bar{X}_n \overset{approx}{\sim} Normal(60, \frac{2^2}{25})$, $\mu = 60$, $\sigma = \frac{2}{\sqrt{25}} = \frac{2}{5} = 0.4$

$P(\bar{X}_n < 59) = P(Z < \frac{59-60}{2/5}) = P(Z < -2.5) = 0.5 - 0.4938 = \boxed{0.0062}$



(e) Suppose the volumes were *not* normally distributed. What is required for the mean volumes of samples of double shots to have approximately a normal distribution?

The sample size must be at least 30 double shots so that the mean volumes of samples of double shots have approximately a normal distribution from the Central Limit Theorem.

3. A botanist sows 100 seeds of a certain type in a greenhouse and is interested in the probability that a seed of this type will germinate under the greenhouse conditions. After some time, she observes that 81 of the 100 seeds germinated. $\hat{p}_n = \frac{81}{100} = 0.81$ 99% CI → $\alpha = 0.01$ proportion

(a) Give an expression for a 99% confidence interval for the probability that this type of seed will germinate under the greenhouse conditions. You do not need to evaluate the bounds of the interval.

$\hat{p}_n = \frac{81}{100} = 0.81$ Wald-Type

$n = 100$

$\alpha = 0.01$

$\alpha/2 = 0.005$

$0.005 \to 2.5758$

$n\hat{p} \geq 15 : 100(0.81) = 81 \geq 15$ ✓

$n(1-\hat{p}) \geq 15 : 100(1-0.81)$

$= 100(0.19)$

$= 19 \geq 15$ ✓

conditions?

$$\hat{p}_n \pm Z_{\alpha/2} \sqrt{\frac{\hat{p}_n(1-\hat{p}_n)}{n}}$$

$$\boxed{0.81 \pm 2.5758 \sqrt{\frac{0.81(1-0.81)}{100}}}$$

(b) The botanist wishes to know if the probability of germination under her greenhouse conditions is more than 0.75. Formulate a null and an alternate hypothesis which correspond to her research question.

interested in if $p > 0.75$

$P(p > 0.75)$

$H_0: p \leq 0.75$

$H_1: p > 0.75$

(c) Write down an expression for the test statistic for testing the hypotheses in the previous part. You do not need to evaluate your expression.

Hypothesis testing for proportion → Ztest statistic

$\hat{p}_n = 0.81$

$p_0 = 0.75$

$n = 100$

$$Z_{test} = \frac{\hat{p}_n - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} = \frac{0.81 - 0.75}{\sqrt{\frac{0.75(1-0.75)}{100}}}$$

(d) The test statistic value is 1.386. Does the researcher reject H_0 at the $\alpha = 0.01$ significance level?

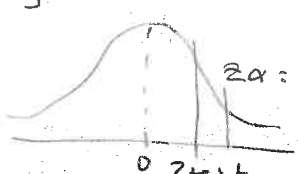
$Z_{test} = 1.386$ Reject H_0 if $Z_{test} > Z_\alpha$

99% CI

$Z_\alpha = Z_{0.01}$ critical value $= 2.3263$

$H_0: p \leq 0.75$

$H_1: p > 0.75$



$Z_{test} < Z_\alpha$

$\boxed{\text{Fail to Reject } H_0 \text{ at } \alpha = 0.01 \text{ significance}}$

(e) Suppose the botanist plans to do a larger study; she wishes in the end to be able to construct a 99% confidence interval with margin of error no greater than two percentage points for the true probability that a seed of this type will germinate under her greenhouse conditions. Give an expression for a guess of the sample size needed for her to achieve this. You do not need to evaluate your expression.

find n sample size for proportion p

CI = 99%

$\alpha = 0.01$

$M = 2\% = 0.02$

$p = 1/2$?

$Z_{\alpha/2} = Z_{0.01/2}$

$= Z_{0.005}$

$= 2.5758$

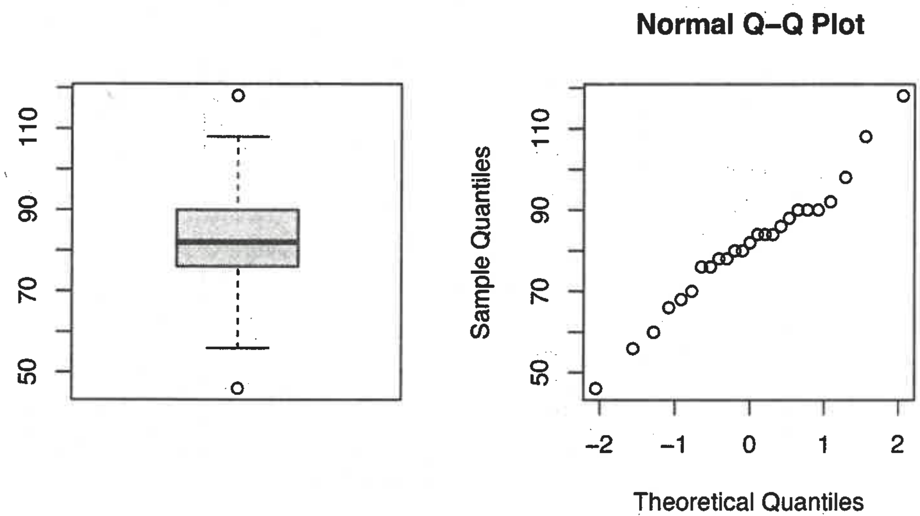
$$n = \left\lceil \left(Z_{\alpha/2} \cdot \frac{\sqrt{p(1-p)}}{M} \right)^2 \right\rceil$$

$$n = \left\lceil \left(2.5758 \cdot \frac{\sqrt{(0.5)(1-0.5)}}{0.02} \right)^2 \right\rceil$$

4. Students in a class were asked to measure their heart rates, resulting in the following set of numbers.

80	84	108	70	84
118	76	90	68	80
92	82	90	46	78
84	76	90	98	66
78	88	86	56	60

The average of all the measurements is 81.12 and the standard deviation is 15.45. A boxplot and a Normal quantile-quantile plot are shown.



(a) Give an expression for a 95% confidence interval for the mean heart rate of this population. You do not have to evaluate the endpoints of your interval.

$\bar{X}_n \pm t_{n-1,\alpha/2} \frac{S_n}{\sqrt{n}}$,

$81.12 \pm 2.0639 \frac{15.45}{5}$

(b) Suppose it is of interest to know whether the mean heart rate in this population differs from 80 bpm. Write down the relevant null and alternate hypotheses.

$H_0 \to \mu = 80$, $H_1 \to \mu \neq 80$

(c) Give an expression for the test statistic for testing the hypotheses in the previous part. You do not have to evaluate your expression.

$$T_{test} = \frac{\bar{X}_n - \mu_0}{S_n/\sqrt{n}} = \frac{81.12 - 80}{15.45/5}$$

(d) The value of the test statistic is 0.362. State your decision about the hypotheses in part $\boxed{b}$ at the $\alpha = 0.05$ significance level.

reject if $|T_{test}| > t_{n-1, \alpha/2} \to$ $.362 < 2.0639$, cannot reject H_0, the mean heart rate does not appear to significantly differ from 80 bpm

(e) Explain the purpose of the Normal quantile-quantile plot.

The normal QQ plot is helpful in seeing if a data's distribution is normally distributed, by checking quantiles of the data distribution to corresponding quantiles of normal distribution. This can help determine which test statistic or conficlence interval to use.