

# STAT 516 Lec 06

## Two-way factorial design (balanced)

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# Tensile strength example from Kuehl (2000)

**Table 6.3** Tensile strength (psi) of asphaltic concrete specimens for two aggregate types with each of four compaction methods

| <i>Aggregate<br/>Type</i> | <i>Compaction Method</i> |                 |            |                 |
|---------------------------|--------------------------|-----------------|------------|-----------------|
|                           | <i>Static</i>            | <i>Kneading</i> |            |                 |
|                           |                          | <i>Regular</i>  | <i>Low</i> | <i>Very Low</i> |
| Basalt                    | 68                       | 126             | 93         | 56              |
|                           | 63                       | 128             | 101        | 59              |
|                           | 65                       | 133             | 98         | 57              |
| Silicious                 | 71                       | 107             | 63         | 40              |
|                           | 66                       | 110             | 60         | 41              |
|                           | 66                       | 116             | 59         | 44              |

*Source:* A. M. Al-Marshed (1981), Compaction effects on asphaltic concrete durability. M.S. thesis, Civil Engineering, University of Arizona.

```
y <- c(68, 63, 65, 71, 66, 66, 126, 128, 133, 107, 110, 116,  
      93, 101, 98, 63, 60, 59, 56, 59, 57, 40, 41, 44)  
agg <- as.factor(rep(c(rep("B", 3), rep("S", 3)), 4))  
comp <- as.factor(c(rep("st", 6), rep("r", 6),  
                    rep("l", 6), rep("vl", 6)))
```

# The two-way factorial experimental design

- ▶ Two factors of interest.
- ▶ Each factor comprehends a set of treatments, called its levels.
- ▶ EUs randomly assigned to treatments.
- ▶ Each treatment is a unique combinations of factor levels.
- ▶ If Factor A has  $a$  levels and Factor B has  $b$  levels.
- ▶ There are  $ab$  treatment groups.

We want to make inferences about:

1. The effects of each factor.
2. The interactions between the two factors.
3. Various differences in treatment group means.

**Discuss:** Give factors and their levels in the tensile strength experiment.

# Main effects and interactions

A main effect is an effect of a factor which does not depend on the level of the other factor.

An interaction is any dependence on the effect of one factor on the level of the other factor.

# Two-way treatment effects model

Suppose the responses arise as

$$Y_{ijk} = \mu + \tau_i + \gamma_j + (\tau\gamma)_{ij} + \varepsilon_{ijk}$$

for  $i = 1, \dots, a$ ,  $j = 1, \dots, b$ , and  $k = 1, \dots, n_{ij}$ , where

- ▶  $Y_{ijk}$  is the response for EU  $k$  under level  $i$  of A and level  $j$  of B.
- ▶  $\mu$  represents a baseline or overall mean
- ▶ The  $\tau_i$  are the main effects for Factor A.
- ▶ The  $\gamma_j$  are the main effects for Factor B.
- ▶ The  $(\tau\gamma)_{ij}$  are the interaction effects between A and B.
- ▶ The  $\varepsilon_{ijk}$  are  $\text{Normal}(0, \sigma^2)$  error terms.

Assume for now

1. a balanced design, i.e.  $n_{ij} = n$  for all  $i, j$
2. with replication, i.e.  $n \geq 2$ .

## Cell means model representation

Assume

$$Y_{ijk} = \mu_{ij} + \varepsilon_{ijk}$$

for  $i = 1, \dots, a$ ,  $j = 1, \dots, b$ , and  $k = 1, \dots, n_{ij}$ , where

- ▶  $Y_{ijk}$  is the response for EU  $k$  under level  $i$  of A and level  $j$  of B.
- ▶  $\mu_{ij}$  is the response mean under level  $i$  of A and level  $j$  of B.
- ▶ The  $\varepsilon_{ijk}$  are  $\text{Normal}(0, \sigma^2)$  error terms.

We have merely written  $\mu_{ij} = \mu + \tau_i + \gamma_j + (\tau\gamma)_{ij}$  for all  $i, j$ .

# Parameter constraints in the treatment effects model

Treatment effects model has  $a + b + ab + 1$  parameters for  $ab$  means...

To identify the parameters uniquely, we usually set

$$\tau_1 = 0, \quad \gamma_1 = 0, \quad \text{and} \quad (\tau\gamma)_{1j} = (\tau\gamma)_{i1} = 0 \text{ for all } i, j.$$

Then  $\mu$  is the mean of the baseline treatment at level 1 of each factor.

## Tensile strength example (cont)

For the tensile strength experiment  $a = 2$  and  $b = 4$ , so the group means

|   | 1          | 2          | 3          | 4          |
|---|------------|------------|------------|------------|
| 1 | $\mu_{11}$ | $\mu_{12}$ | $\mu_{13}$ | $\mu_{14}$ |
| 2 | $\mu_{21}$ | $\mu_{22}$ | $\mu_{23}$ | $\mu_{24}$ |

are parameterized under the  $\mu$ -as-baseline constraint as:

|   | 1              | 2   | 3   | 4   |
|---|----------------|---|---|---|
| 1 | $\mu$          | $\mu + \gamma_2$                              | $\mu + \gamma_3$                              | $\mu + \gamma_4$                              |
| 2 | $\mu + \tau_2$ | $\mu + \tau_2 + \gamma_2 + (\tau\gamma)_{22}$ | $\mu + \tau_2 + \gamma_3 + (\tau\gamma)_{23}$ | $\mu + \tau_2 + \gamma_4 + (\tau\gamma)_{24}$ |

# Marginal means in the two-way factorial experiment

Define the marginal means and the overall mean as

$$\blacktriangleright \bar{\mu}_{i.} = \frac{1}{b} \sum_{j=1}^b \mu_{ij}, \quad i = 1, \dots, a.$$

$$\blacktriangleright \bar{\mu}_{.j} = \frac{1}{a} \sum_{i=1}^a \mu_{ij}, \quad j = 1, \dots, b.$$

$$\blacktriangleright \bar{\mu}_{..} = \frac{1}{ab} \sum_{i=1}^a \sum_{j=1}^b \mu_{ij}.$$

# Hypotheses of interest in the two-way factorial experiment

1.  $H_0$ : Factor A has no main effect.

$$H_0: \bar{\mu}_{1.} = \cdots = \bar{\mu}_{a.}$$

2.  $H_0$ : Factor B has no main effect.

$$H_0: \bar{\mu}_{.1} = \cdots = \bar{\mu}_{.b}$$

3.  $H_0$ : There is no interaction between Factor A and Factor B.

$$H_0: \mu_{ij} = \bar{\mu}_{i.} + \bar{\mu}_{.j} - \bar{\mu}_{..} \quad \text{for all } i, j.$$

(Factor A effects are same at all levels of Factor B and vice versa.)

**Example:** Let  $a = 2$  and  $b = 4$ . Can tabulate the response means as:

|   | 1          | 2          | 3          | 4          |
|---|------------|------------|------------|------------|
| 1 | $\mu_{11}$ | $\mu_{12}$ | $\mu_{13}$ | $\mu_{14}$ |
| 2 | $\mu_{21}$ | $\mu_{22}$ | $\mu_{23}$ | $\mu_{24}$ |

Use table to discuss hypotheses of no main effects and of no interaction.

# Goals in two-way factorial experiments

In the two-way treatment effects model

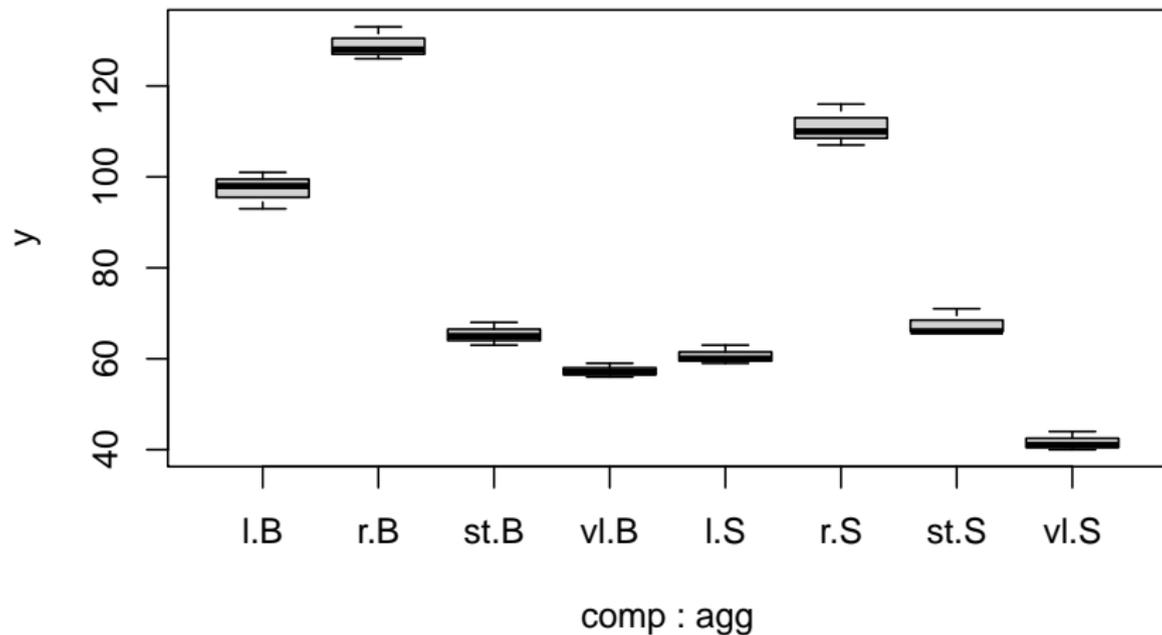
$$Y_{ijk} = \mu + \tau_i + \gamma_j + (\tau\gamma)_{ij} + \varepsilon_{ijk},$$

where  $\varepsilon_{ijk} \stackrel{\text{ind}}{\sim} \text{Normal}(0, \sigma^2)$ , we wish to

1. Visualize the data.
2. Estimate the parameters  $\mu, \tau_i, \gamma_j, (\tau\gamma)_{ij}$ .
3. Estimate the error term variance  $\sigma^2$ .
4. Check whether the model assumptions are satisfied.
5. Decompose the variation in the  $Y_{ijk}$  into its sources.
6. Test whether there is *any* effect of the factors on the response.
7. Test for main effects and interaction effects.
8. Do multiple comparisons.

# Tensile strength data (cont)

```
boxplot(y ~ comp:agg)
```



# Tensile example (cont)

Use `summary()` on the `lm()` function output.

```
lm_out <- lm(y ~ agg + comp + agg:comp)
summary(lm_out)
```

Call:

```
lm(formula = y ~ agg + comp + agg:comp)
```

Residuals:

| Min     | 1Q      | Median  | 3Q     | Max    |
|---------|---------|---------|--------|--------|
| -4.3333 | -1.6667 | -0.6667 | 2.3333 | 5.0000 |

Coefficients:

|             | Estimate | Std. Error | t value | Pr(> t )     |
|-------------|----------|------------|---------|--------------|
| (Intercept) | 97.333   | 1.780      | 54.697  | < 2e-16 ***  |
| aggS        | -36.667  | 2.517      | -14.570 | 1.18e-10 *** |
| compr       | 31.667   | 2.517      | 12.583  | 1.03e-09 *** |
| compst      | -32.000  | 2.517      | -12.716 | 8.85e-10 *** |
| compvl      | -40.000  | 2.517      | -15.894 | 3.20e-11 *** |
| aggS:compr  | 18.667   | 3.559      | 5.245   | 8.01e-05 *** |
| aggS:compst | 39.000   | 3.559      | 10.958  | 7.58e-09 *** |
| aggS:compvl | 21.000   | 3.559      | 5.900   | 2.24e-05 *** |

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Residual standard error: 3.082 on 16 degrees of freedom

Multiple R-squared: 0.9921, Adjusted R-squared: 0.9887

F-statistic: 287.6 on 7 and 16 DF, p-value: 1.305e-15

**Exercise:** See how the parameter estimates build the treatment means.

```
aggregate(y ~ agg + comp, FUN = mean)
```

|   | agg | comp | y         |
|---|-----|------|-----------|
| 1 | B   | l    | 97.33333  |
| 2 | S   | l    | 60.66667  |
| 3 | B   | r    | 129.00000 |
| 4 | S   | r    | 111.00000 |
| 5 | B   | st   | 65.33333  |
| 6 | S   | st   | 67.66667  |
| 7 | B   | v1   | 57.33333  |
| 8 | S   | v1   | 41.66667  |

## The fitted values

Define the treatment group means

$$\bar{Y}_{ij.} = \frac{1}{n} \sum_{k=1}^n Y_{ijk} \quad \text{for } i = 1, \dots, a, \quad j = 1, \dots, b.$$

Then the

- ▶ fitted values are the treatment group means, i.e.  $\hat{Y}_{ijk} = \bar{Y}_{ij.}$
- ▶ residuals are the deviations from the group means  $\varepsilon_{ijk} = Y_{ijk} - \bar{Y}_{ij.}$

In the cell means model, we estimate  $\mu_{ij}$  with  $\hat{\mu}_{ij} = \bar{Y}_{ij.} \quad \forall ij.$

## Estimating the error term variance

An unbiased estimator of the error term variance  $\sigma^2$  is given by

$$\hat{\sigma}^2 = \frac{1}{ab(n-1)} \sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^n (Y_{ijk} - \bar{Y}_{ij.})^2,$$

which is the sum of the squared residuals divided by  $ab(n-1)$ .

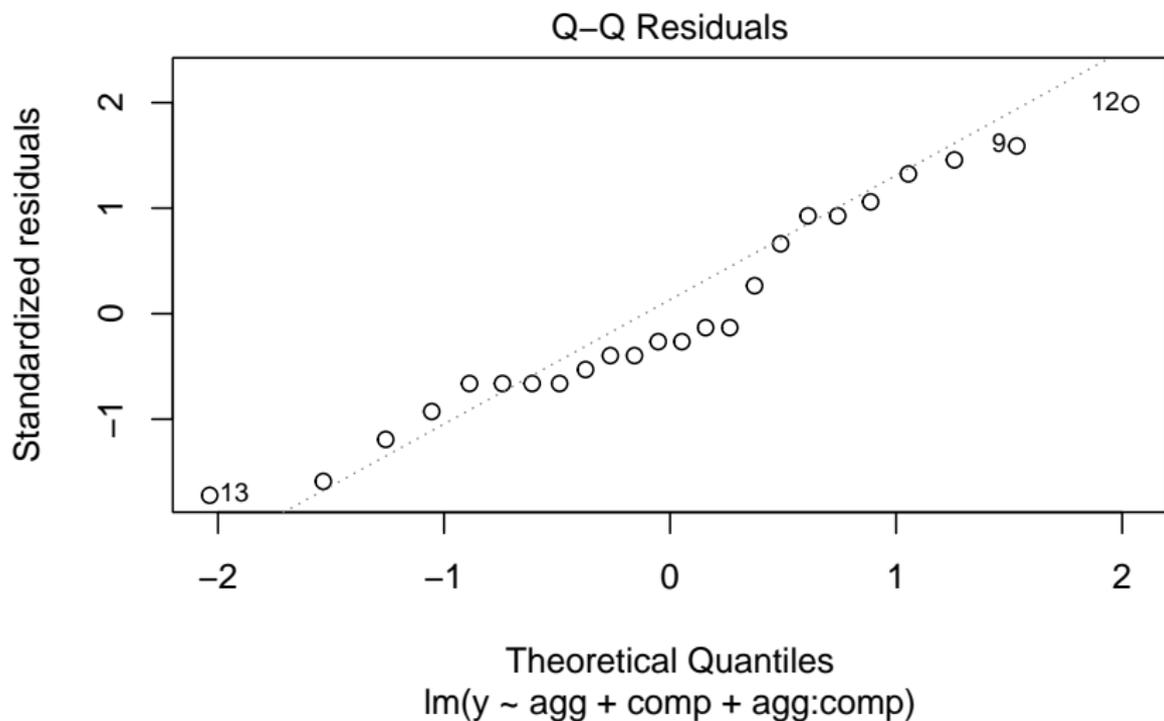
# Checking model assumptions

Validity of the methods in these slides depends on these assumptions:

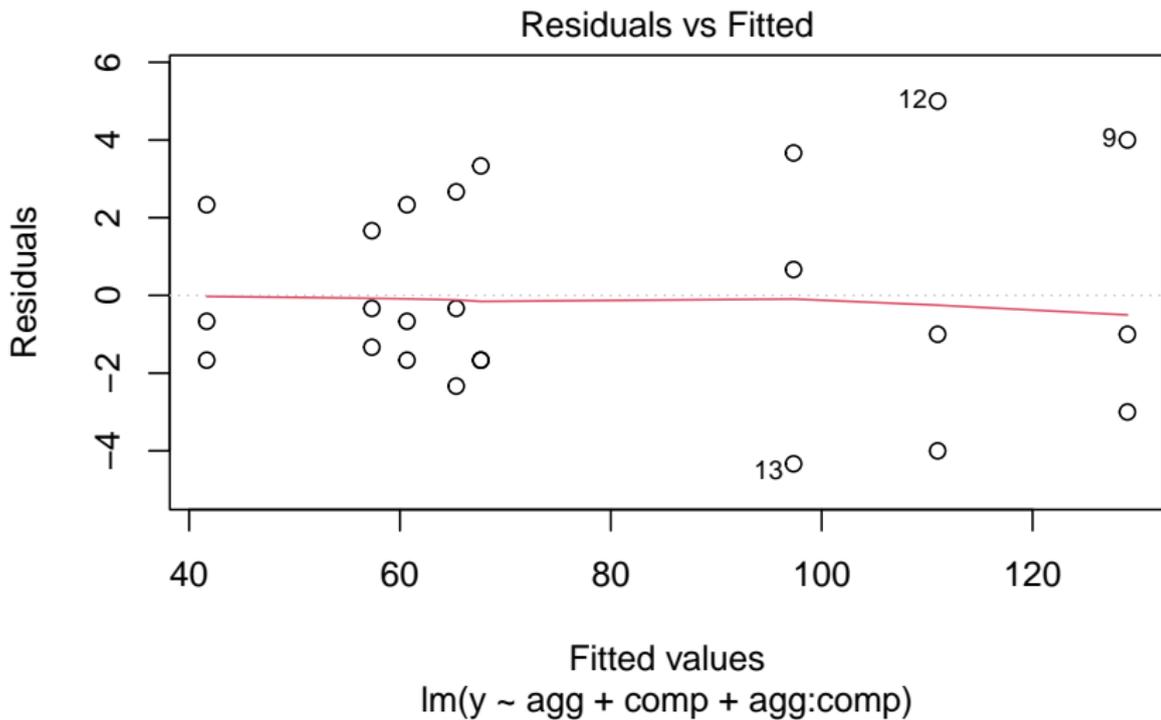
1. The responses are normally distributed around the treatment means (Check QQ plot of residuals).
2. The response has the same variance in all treatment groups (Check residuals vs fitted values plot).
3. The response values are independent of each other (No way to check; must trust experimental design).

## Tensile example (cont)

```
plot(lm_out, which = 2)
```



```
plot(lm_out, which = 1)
```



## Sums of squares in the two-way factorial experiment

| SS         | Symbol              | Formula  |
|------------|---------------------|--|
| Total      | $SS_{\text{Tot}}$   | $\sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^n (Y_{ijk} - \bar{Y}_{...})^2$ |
| Treatments | $SS_{\text{Trt}}$   | $\sum_{i=1}^a \sum_{j=1}^b n(\bar{Y}_{ij.} - \bar{Y}_{...})^2$       |
| Error      | $SS_{\text{Error}}$ | $\sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^n (Y_{ijk} - \bar{Y}_{ij.})^2$ |

- ▶ We have  $SS_{\text{Tot}} = SS_{\text{Trt}} + SS_{\text{Error}}$ .
- ▶ We again define  $R^2 = \frac{SS_{\text{Trt}}}{SS_{\text{Tot}}}$ .
- ▶ Can further decompose  $SS_{\text{Trt}}$  into SS from main effects and interactions.

## ANOVA table for overall F test

| Source     | Df          | SS                  | MS                  | F value           | p value                  |
|------------|-------------|---------------------|---------------------|-------------------|--------------------------|
| Treatments | $ab - 1$    | $SS_{\text{Trt}}$   | $MS_{\text{Trt}}$   | $F_{\text{stat}}$ | $P(F > F_{\text{stat}})$ |
| Error      | $ab(n - 1)$ | $SS_{\text{Error}}$ | $MS_{\text{Error}}$ |                   |                          |
| Total      | $abn - 1$   | $SS_{\text{Tot}}$   |                     |                   |                          |

▶ The F statistic  $F_{\text{stat}} = \frac{MS_{\text{Trt}}}{MS_{\text{Error}}}$  tests

$H_0$ : All the  $\mu_{ij}$  are the same.

$H_1$ : The  $\mu_{ij}$  are not all the same.

▶ Reject  $H_0$  if  $F_{\text{stat}} > F_{ab-1, ab(n-1), \alpha}$ .

**Exercise:** Make ANOVA table for the tensile strength data. Interpret!

```
a <- 2
b <- 4
n <- 3
yhat <- predict(lm_out)
ehat <- y - yhat
SSE <- sum(ehat^2)
MSE <- SSE / (a*b*(n-1))
SSR <- sum((yhat - mean(y))^2)
MSR <- SSR / (a*b - 1)
SST <- sum((y - mean(y))^2)
Fstat <- MSR / MSE
pval <- 1 - pf(Fstat, a*b-1, a*b*(n-1))
```

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| Source     | Df | SS       | MS      | F value  | p value |
|------------|----|----------|---------|----------|---------|
| Treatments | 7  | 19122.50 | 2731.79 | 287.5564 | 0.0000  |
| Error      | 16 | 152.00   | 9.50    |          |         |
| Total      | 23 | 19274.50 |         |          |         |

---

## Further decomposition of treatments sum of squares

Define main effect and interaction sums of squares  $SS_A$ ,  $SS_B$ , and  $SS_{AB}$ :

$$\blacktriangleright SS_A = bn \sum_{i=1}^a (\bar{Y}_{i..} - \bar{Y}_{...})^2$$

$$\blacktriangleright SS_B = an \sum_{j=1}^b (\bar{Y}_{.j.} - \bar{Y}_{...})^2$$

$$\blacktriangleright SS_{AB} = n \sum_{i=1}^a \sum_{j=1}^b (\bar{Y}_{ij.} - (\bar{Y}_{i..} + \bar{Y}_{.j.} - \bar{Y}_{...}))^2$$

In the balanced design we have

$$SS_{\text{Trt}} = SS_A + SS_B + SS_{AB} .$$

## Full sums of squares in two-way factorial experiment

| SS    | Symbol              | Formula   |
|-------|---------------------|---|
| Total | $SS_{\text{Tot}}$   | $\sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^n (Y_{ijk} - \bar{Y}_{...})^2$                        |
| A     | $SS_A$              | $bn \sum_{i=1}^a (\bar{Y}_{i..} - \bar{Y}_{...})^2$   |
| B     | $SS_B$              | $an \sum_{j=1}^b (\bar{Y}_{.j.} - \bar{Y}_{...})^2$   |
| AB    | $SS_{AB}$           | $n \sum_{i=1}^a \sum_{j=1}^b (Y_{ij.} - (\bar{Y}_{i..} + \bar{Y}_{.j.} - \bar{Y}_{...}))^2$ |
| Error | $SS_{\text{Error}}$ | $\sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^n (Y_{ijk} - \bar{Y}_{ij.})^2$                        |

► We have  $SS_{\text{Tot}} = SS_A + SS_B + SS_{AB} + SS_{\text{Error}}$ .

## Full ANOVA table for balanced two-way factorial design

| Source | Df               | SS                  | MS                  | F value                                |
|--------|------------------|---------------------|---------------------|--|
| A      | $a - 1$          | $SS_A$              | $MS_A$              | $F_A = MS_A / MS_{\text{Error}}$       |
| B      | $b - 1$          | $SS_B$              | $MS_B$              | $F_B = MS_B / MS_{\text{Error}}$       |
| AB     | $(a - 1)(b - 1)$ | $SS_{AB}$           | $MS_{AB}$           | $F_{AB} = MS_{AB} / MS_{\text{Error}}$ |
| Error  | $ab(n - 1)$      | $SS_{\text{Error}}$ | $MS_{\text{Error}}$ |  |
| Total  | $abn - 1$        | $SS_{\text{Tot}}$   |                     |  |

1. Reject  $H_0$ : no Factor A main effect if  $F_A > F_{a-1, ab(n-1), \alpha}$ .
2. Reject  $H_0$ : no Factor B main effect if  $F_B > F_{b-1, ab(n-1), \alpha}$ .
3. Reject  $H_0$ : no A and B interaction if  $F_{AB} > F_{(a-1)(b-1), ab(n-1), \alpha}$ .

## Tensile strength data (cont)

Obtain ANOVA table with `anova()` function on `lm()` output.

```
anova(lm(y ~ agg + comp + agg:comp))
```

Analysis of Variance Table

Response: y

|           | Df | Sum Sq | Mean Sq | F value | Pr(>F)    |     |
|-----------|----|--------|---------|---------|-----------|-----|
| agg       | 1  | 1734   | 1734.0  | 182.526 | 3.628e-10 | *** |
| comp      | 3  | 16244  | 5414.5  | 569.947 | < 2.2e-16 | *** |
| agg:comp  | 3  | 1145   | 381.7   | 40.175  | 1.124e-07 | *** |
| Residuals | 16 | 152    | 9.5     |         |           |     |

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**Important:** `anova()` function only appropriate for a balanced design.

## Interaction is significant. Now what?

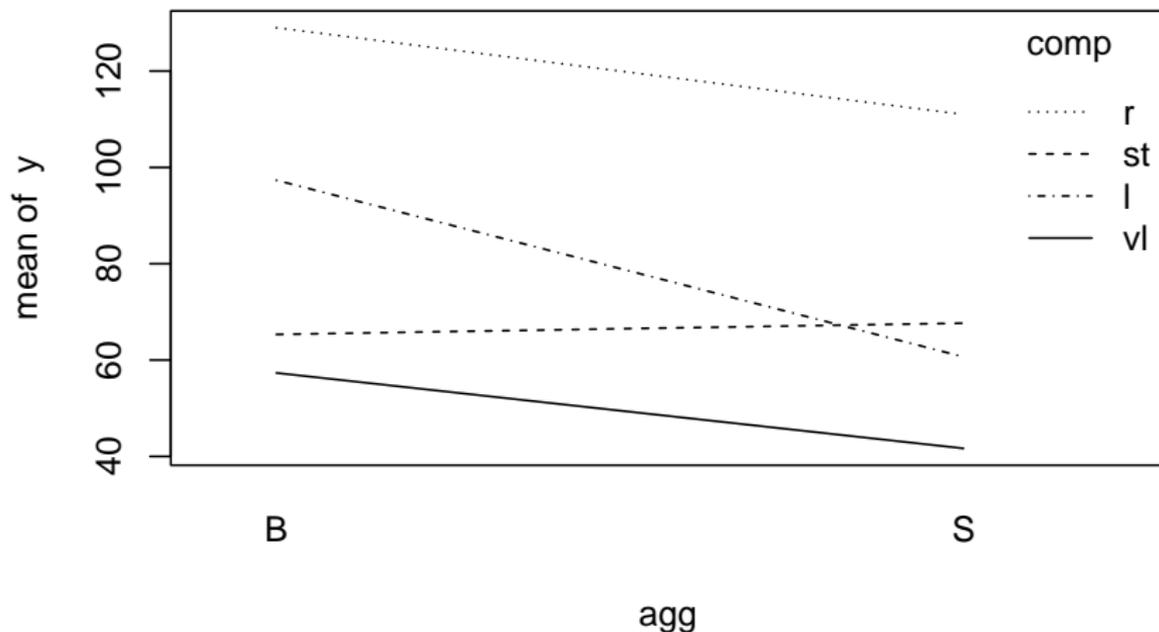
When you find a significant interaction:

1. Make interaction plots (next slides).
2. Be very cautious about interpreting main effects, even when these are statistically significant.

## Tensile strength data (cont)

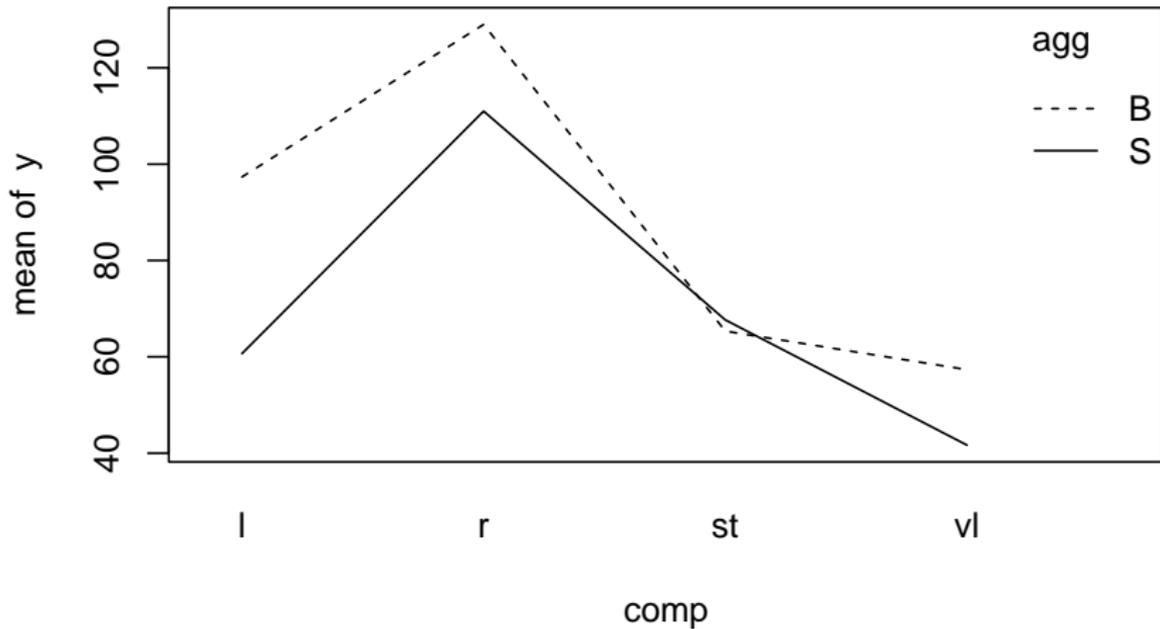
Use the `interaction.plot()` function to visualize an interaction:

```
interaction.plot(agg, comp, y)
```



Interactions appear as crossing lines or differing slopes.

```
interaction.plot(comp, agg, y)
```



# Estimates of cell and marginal means in the balanced case

The estimators of the cell and marginal means are given by

▶  $\hat{\mu}_{ij} = \bar{Y}_{ij}, i = 1, \dots, a, j = 1, \dots, b$

▶  $\hat{\mu}_{i.} = \bar{Y}_{i.}, i = 1, \dots, a.$

▶  $\hat{\mu}_{.j} = \bar{Y}_{.j}, j = 1, \dots, b.$

We estimate  $\hat{\mu}_{i.}$  with  $\bar{Y}_{i.}$  (and  $\hat{\mu}_{.j}$  with  $\bar{Y}_{.j}.$ ) only when  $n_{ij} = n \forall ij.$

## Some CI formulas (without familywise adjustment)

These CI formulas are for the balanced design  $n_{ij} = n \forall ij$ .

| Target                              | $(1 - \alpha)100\%$ confidence interval   |
|-------------------------------------|---|
| $\mu_{ij}$                          | $\bar{Y}_{ij} \pm t_{ab(n-1), \alpha/2} \hat{\sigma} \sqrt{\frac{1}{n}}$                    |
| $\mu_{ij} - \mu_{i'j'}$             | $\bar{Y}_{ij} - \bar{Y}_{i'j'} \pm t_{ab(n-1), \alpha/2} \hat{\sigma} \sqrt{\frac{2}{n}}$   |
| $\bar{\mu}_{i.}$                    | $\bar{Y}_{i..} \pm t_{ab(n-1), \alpha/2} \hat{\sigma} \sqrt{\frac{1}{bn}}$                  |
| $\bar{\mu}_{.j}$                    | $\bar{Y}_{.j.} \pm t_{ab(n-1), \alpha/2} \hat{\sigma} \sqrt{\frac{1}{an}}$                  |
| $\bar{\mu}_{i.} - \bar{\mu}_{i'..}$ | $\bar{Y}_{i..} - \bar{Y}_{i'..} \pm t_{ab(n-1), \alpha/2} \hat{\sigma} \sqrt{\frac{2}{bn}}$ |
| $\bar{\mu}_{.j} - \bar{\mu}_{.j'}$  | $\bar{Y}_{.j.} - \bar{Y}_{.j'.} \pm t_{ab(n-1), \alpha/2} \hat{\sigma} \sqrt{\frac{2}{an}}$ |

In the above  $\hat{\sigma} = \sqrt{MS_{\text{Error}}}$ .

## Comparing means at all factor level combinations

- ▶ Tukey's for comparing all pairs among  $\mu_{ij}$ ,  $i = 1, \dots, a$ ,  $j = 1, \dots, b$ :

$$\bar{Y}_{ij.} - \bar{Y}_{i'j'}. \pm q_{ab,ab(n-1),\alpha} \hat{\sigma} \frac{1}{\sqrt{n}}, \quad (i, j) \neq (i', j').$$

- ▶ Dunnett's for comparing all means  $\mu_{ij}$  to a baseline  $\mu_{ab}$ :

$$\bar{Y}_{ij.} - \bar{Y}_{ab.} \pm d_{ab,ab(n-1),\alpha} \hat{\sigma} \sqrt{\frac{2}{n}}, \quad (i, j) \neq (a, b).$$

Use  $\hat{\sigma} = \sqrt{MS_{\text{Error}}}$ .

These may make more comparisons than are of interest...

# Tensile strength data (cont)

```
TukeyHSD(aov(lm(y ~ agg:comp)))
```

Tukey multiple comparisons of means  
95% family-wise confidence level

```
Fit: aov(formula = lm(y ~ agg:comp))
```

```
$`agg:comp`  
      diff      lwr      upr      p adj  
S:l-B:l -36.666667 -45.379555 -27.9537785 0.0000000  
B:r-B:l  31.666667  22.953778  40.3795548 0.0000000  
S:r-B:l  13.666667  4.953778  22.3795548 0.0011210  
B:st-B:l -32.000000 -40.712888 -23.2871118 0.0000000  
S:st-B:l -29.666667 -38.379555 -20.9537785 0.0000001  
B:vl-B:l -40.000000 -48.712888 -31.2871118 0.0000000  
S:vl-B:l -55.666667 -64.379555 -46.9537785 0.0000000  
B:r-S:l  68.333333  59.620445  77.0462215 0.0000000  
S:r-S:l  50.333333  41.620445  59.0462215 0.0000000  
B:st-S:l  4.666667  -4.046222  13.3795548 0.5964603  
S:st-S:l  7.000000  -1.712888  15.7128882 0.1678762  
B:vl-S:l -3.333333 -12.046222  5.3795548 0.8765993  
S:vl-S:l -19.000000 -27.712888 -10.2871118 0.0000257  
S:r-B:r -18.000000 -26.712888 -9.2871118 0.0000501  
B:st-B:r -63.666667 -72.379555 -54.9537785 0.0000000  
S:st-B:r -61.333333 -70.046222 -52.6204452 0.0000000  
B:vl-B:r -71.666667 -80.379555 -62.9537785 0.0000000  
S:vl-B:r -87.333333 -96.046222 -78.6204452 0.0000000  
B:st-S:r -45.666667 -54.379555 -36.9537785 0.0000000  
S:st-S:r -43.333333 -52.046222 -34.6204452 0.0000000  
B:vl-S:r -53.666667 -62.379555 -44.9537785 0.0000000  
S:vl-S:r -69.333333 -78.046222 -60.6204452 0.0000000  
S:st-B:st  2.333333  -6.379555  11.0462215 0.9785200  
B:vl-B:st -8.000000 -16.712888  0.7128882 0.0842128  
S:vl-B:st -23.666667 -32.379555 -14.9537785 0.0000015  
B:vl-S:st -10.333333 -19.046222 -1.6204452 0.0145554  
S:vl-S:st -26.000000 -34.712888 -17.2871118 0.0000004  
S:vl-B:vl -15.666667 -24.379555 -6.9537785 0.0002561
```

Easiest way to do Dunnett's is to convert the design to a one-way:

```
agg_comp <- as.factor(paste(agg,comp,sep="_"))
levels(agg_comp)
```

```
[1] "B_l" "B_r" "B_st" "B_vl" "S_l" "S_r" "S_st" "S_vl"
```

```
library(DescTools)
DunnettTest(y ~ agg_comp, control = "B_st", conf.level = 0.95)
```

Dunnett's test for comparing several treatments with a control :  
95% family-wise confidence level

```
$B_st
      diff      lwr.ci      upr.ci      pval
B_l-B_st 32.000000 24.643829 39.3561715 1.2e-09 ***
B_r-B_st 63.666667 56.310495 71.0228381 < 2e-16 ***
B_vl-B_st -8.000000 -15.356171 -0.6438285 0.0304 *
S_l-B_st -4.666667 -12.022838  2.6895048 0.3266
S_r-B_st 45.666667 38.310495 53.0228381 < 2e-16 ***
S_st-B_st  2.333333 -5.022838  9.6895048 0.8881
S_vl-B_st -23.666667 -31.022838 -16.3104952 4.4e-08 ***
```

---

```
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

## Comparing factor level means at fixed level of other factor

Fix Factor A at level  $i$  and compare means across factor B:

- ▶ Tukey's for comparing all pairs among  $\bar{\mu}_{i1}, \dots, \bar{\mu}_{ib}$ :

$$\bar{Y}_{ij.} - \bar{Y}_{ij'.} \pm q_{b,ab(n-1),\alpha} \hat{\sigma} \frac{1}{\sqrt{n}}, \quad 1 \leq j < j' \leq b.$$

- ▶ Dunnett's for comparing  $\bar{\mu}_{ij}, \dots, \bar{\mu}_{i,b-1}$  to baseline  $\mu_{ib}$ :

$$\bar{Y}_{ij.} - \bar{Y}_{ib.} \pm d_{b,ab(n-1),\alpha} \hat{\sigma} \sqrt{\frac{2}{n}}, \quad j = 1, \dots, b-1.$$

To do this at *all* levels  $i = 1, \dots, a$ , divide  $\alpha$  by  $a$  (à la Bonferroni)!

Likewise if comparing means of factor B at fixed levels of factor A.

## Tensile strength data (cont)

For each aggregate type, compare all pairs of compaction method means.

Ensure that the familywise coverage probability is at least 0.95.

Use  $q_{4,16,0.05/2} = \text{qtukey}(1-0.05/2, 4, 16) = 4.54763$ .

```
y11. <- mean(y[agg == "B" & comp == "l"])
y12. <- mean(y[agg == "B" & comp == "r"])
y13. <- mean(y[agg == "B" & comp == "st"])
y14. <- mean(y[agg == "B" & comp == "v1"])

y21. <- mean(y[agg == "S" & comp == "l"])
y22. <- mean(y[agg == "S" & comp == "r"])
y23. <- mean(y[agg == "S" & comp == "st"])
y24. <- mean(y[agg == "S" & comp == "v1"])

alpha <- 0.05
me <- qtkey(1-alpha/a,b,a*b*(n-1)) * sqrt(MSE) / sqrt(n)
```

```

ttab <- rbind(c(y11. - y12. - me, y11. - y12. + me),
             c(y11. - y13. - me, y11. - y13. + me),
             c(y11. - y14. - me, y11. - y14. + me),
             c(y12. - y13. - me, y12. - y13. + me),
             c(y12. - y14. - me, y12. - y14. + me),
             c(y13. - y14. - me, y13. - y14. + me),
             c(y21. - y22. - me, y21. - y22. + me),
             c(y21. - y23. - me, y21. - y23. + me),
             c(y21. - y24. - me, y21. - y24. + me),
             c(y22. - y23. - me, y22. - y23. + me),
             c(y22. - y24. - me, y22. - y24. + me),
             c(y23. - y24. - me, y23. - y24. + me))
rownames(ttab) <- c("B:l-r", "B:l-st", "B:l-vl", "B:r-st", "B:r-vl", "B:st-vl",
                  "S:l-r", "S:l-st", "S:l-vl", "S:r-st", "S:r-vl", "S:st-vl")
colnames(ttab) <- c("lower", "upper")

```

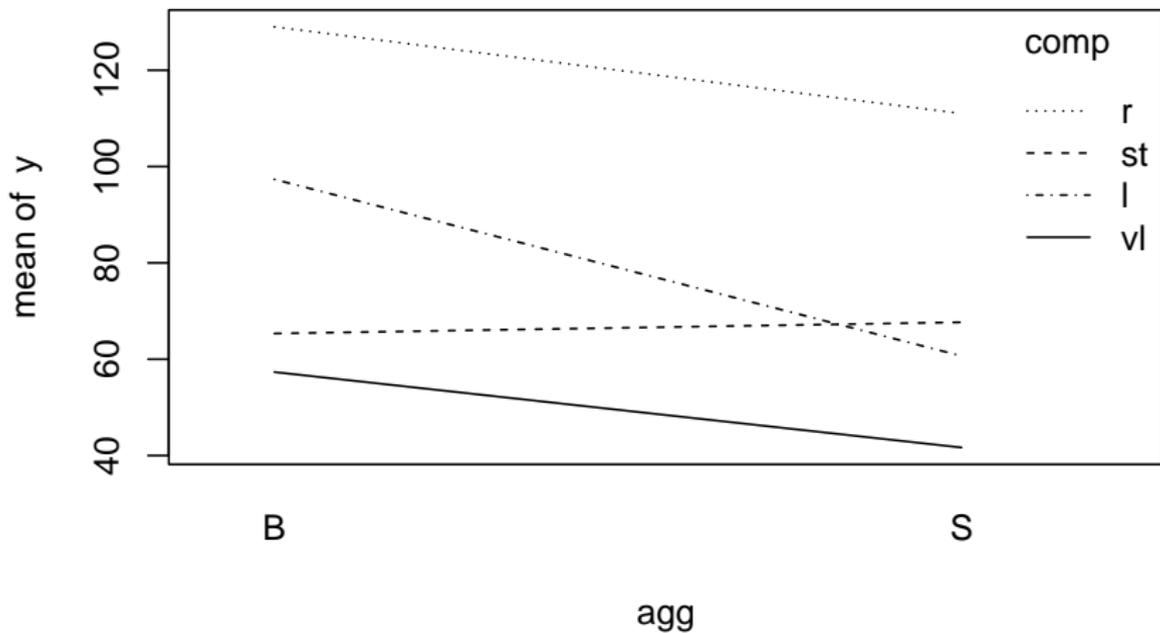
Tukey/Bonferroni-adjusted confidence intervals comparing all pairs of compaction methods for each aggregate type:

```
ttab
```

|         | lower        | upper      |
|---------|--------------|------------|
| B:l-r   | -39.75923299 | -23.574100 |
| B:l-st  | 23.90743367  | 40.092566  |
| B:l-vl  | 31.90743367  | 48.092566  |
| B:r-st  | 55.57410034  | 71.759233  |
| B:r-vl  | 63.57410034  | 79.759233  |
| B:st-vl | -0.09256633  | 16.092566  |
| S:l-r   | -58.42589966 | -42.240767 |
| S:l-st  | -15.09256633 | 1.092566   |
| S:l-vl  | 10.90743367  | 27.092566  |
| S:r-st  | 35.24076701  | 51.425900  |
| S:r-vl  | 61.24076701  | 77.425900  |
| S:st-vl | 17.90743367  | 34.092566  |

Identify which differences are *not* significant in the interaction plot:

```
interaction.plot(agg, comp, y)
```



## Tensile strength data (cont)

For only the basalt aggregate type, compare all compaction method means to that of the static method.

Use  $d_{4,16,0.05} = 2.59$  from the table on the next slide.

```
me <- 2.59 * sqrt(MSE) * sqrt(2/n)
dtab <- rbind(c(y11. - y13. - me, y11. - y13. + me),
             c(y12. - y13. - me, y12. - y13. + me),
             c(y14. - y13. - me, y14. - y13. + me))
rownames(dtab) <- c("B:l-st", "B:r-st", "B:v1-st")
colnames(dtab) <- c("lower", "upper")
```

Table A.5 Critical Values for Dunnett's Two-Sided Test of Treatments versus Control.

| Error df | Two-sided $\alpha$ | T = Number of Groups Counting Both Treatments and Control |      |      |      |      |      |      |
|----------|--------------------|---|------|------|------|------|------|------|
|          |                    | 2   | 3    | 4    | 5    | 6    | 7    | 8    |
| 5        | 0.05               | 2.57  | 3.03 | 3.29 | 3.48 | 3.62 | 3.73 | 3.82 |
| 5        | 0.01               | 4.03  | 4.63 | 4.97 | 5.22 | 5.41 | 5.56 | 5.68 |
| 6        | 0.05               | 2.45  | 2.86 | 3.10 | 3.26 | 3.39 | 3.49 | 3.57 |
| 6        | 0.01               | 3.71  | 4.21 | 4.51 | 4.71 | 4.87 | 5.00 | 5.10 |
| 7        | 0.05               | 2.36  | 2.75 | 2.97 | 3.12 | 3.24 | 3.33 | 3.41 |
| 7        | 0.01               | 3.50  | 3.95 | 4.21 | 4.39 | 4.53 | 4.64 | 4.74 |
| 8        | 0.05               | 2.31  | 2.67 | 2.88 | 3.02 | 3.13 | 3.22 | 3.29 |
| 8        | 0.01               | 3.36  | 3.77 | 4.00 | 4.17 | 4.29 | 4.40 | 4.48 |
| 9        | 0.05               | 2.26  | 2.61 | 2.81 | 2.95 | 3.05 | 3.14 | 3.20 |
| 9        | 0.01               | 3.25  | 3.63 | 3.85 | 4.01 | 4.12 | 4.22 | 4.30 |
| 10       | 0.05               | 2.23  | 2.57 | 2.76 | 2.89 | 2.99 | 3.07 | 3.14 |
| 10       | 0.01               | 3.17  | 3.53 | 3.74 | 3.88 | 3.99 | 4.08 | 4.16 |
| 11       | 0.05               | 2.20  | 2.53 | 2.72 | 2.84 | 2.94 | 3.02 | 3.08 |
| 11       | 0.01               | 3.11  | 3.45 | 3.65 | 3.79 | 3.89 | 3.98 | 4.05 |
| 12       | 0.05               | 2.18  | 2.50 | 2.68 | 2.81 | 2.90 | 2.98 | 3.04 |
| 12       | 0.01               | 3.05  | 3.39 | 3.58 | 3.71 | 3.81 | 3.89 | 3.96 |
| 13       | 0.05               | 2.16  | 2.48 | 2.65 | 2.78 | 2.87 | 2.94 | 3.00 |
| 13       | 0.01               | 3.01  | 3.33 | 3.52 | 3.65 | 3.74 | 3.82 | 3.89 |
| 14       | 0.05               | 2.14  | 2.46 | 2.63 | 2.75 | 2.84 | 2.91 | 2.97 |
| 14       | 0.01               | 2.98  | 3.29 | 3.47 | 3.59 | 3.69 | 3.76 | 3.83 |
| 15       | 0.05               | 2.13  | 2.44 | 2.61 | 2.73 | 2.82 | 2.89 | 2.95 |
| 15       | 0.01               | 2.95  | 3.25 | 3.43 | 3.55 | 3.64 | 3.71 | 3.78 |
| 16       | 0.05               | 2.12  | 2.42 | 2.59 | 2.71 | 2.80 | 2.87 | 2.92 |
| 16       | 0.01               | 2.92  | 3.22 | 3.39 | 3.51 | 3.60 | 3.67 | 3.73 |
| 17       | 0.05               | 2.11  | 2.41 | 2.58 | 2.69 | 2.78 | 2.85 | 2.90 |
| 17       | 0.01               | 2.90  | 3.19 | 3.36 | 3.47 | 3.56 | 3.63 | 3.69 |
| 18       | 0.05               | 2.10  | 2.40 | 2.56 | 2.68 | 2.76 | 2.83 | 2.89 |
| 18       | 0.01               | 2.88  | 3.17 | 3.33 | 3.44 | 3.53 | 3.60 | 3.66 |
| 19       | 0.05               | 2.09  | 2.39 | 2.55 | 2.66 | 2.75 | 2.81 | 2.87 |
| 19       | 0.01               | 2.86  | 3.15 | 3.31 | 3.42 | 3.50 | 3.57 | 3.63 |
| 20       | 0.05               | 2.09  | 2.38 | 2.54 | 2.65 | 2.73 | 2.80 | 2.86 |
| 20       | 0.01               | 2.85  | 3.13 | 3.29 | 3.40 | 3.48 | 3.55 | 3.60 |
| 25       | 0.05               | 2.06  | 2.34 | 2.50 | 2.61 | 2.69 | 2.75 | 2.81 |
| 25       | 0.01               | 2.79  | 3.06 | 3.21 | 3.31 | 3.39 | 3.45 | 3.51 |
| 30       | 0.05               | 2.04  | 2.32 | 2.47 | 2.58 | 2.66 | 2.72 | 2.77 |
| 30       | 0.01               | 2.75  | 3.01 | 3.15 | 3.25 | 3.33 | 3.39 | 3.44 |
| 40       | 0.05               | 2.02  | 2.29 | 2.44 | 2.54 | 2.62 | 2.68 | 2.73 |
| 40       | 0.01               | 2.70  | 2.95 | 3.09 | 3.19 | 3.26 | 3.32 | 3.37 |
| 60       | 0.05               | 2.00  | 2.27 | 2.41 | 2.51 | 2.58 | 2.64 | 2.69 |
| 60       | 0.01               | 2.66  | 2.90 | 3.03 | 3.12 | 3.19 | 3.25 | 3.29 |

This table produced from the SAS System using function PROCBMCT(DUNNETT2, 1 -  $\alpha$ , df, k), where  $k = T - 1$ .

Figure 1: Table A.5 from Mohr, Wilson, and Freund (2021)

Dunnett's comparison of compaction method means to the static method when the aggregate type is basalt:

```
dtab
```

|         | lower     | upper     |
|---------|-----------|-----------|
| B:l-st  | 25.48198  | 38.518024 |
| B:r-st  | 57.14864  | 70.184690 |
| B:vl-st | -14.51802 | -1.481976 |

## Interaction *not* significant. Then what?

If the interaction is not significant:

1. We can interpret main effects.
2. We can make meaningful comparisons among marginal means.

## Comparing marginal means in the absence of interaction

For making comparisons among the marginal means of Factor A:

- ▶ Tukey's for comparing all pairs among  $\bar{\mu}_{1.}, \dots, \bar{\mu}_{a.}$ :

$$\bar{Y}_{i..} - \bar{Y}_{i'..} \pm q_{a,ab(n-1),\alpha} \hat{\sigma} \frac{1}{\sqrt{bn}}, \quad 1 \leq i < i' \leq a.$$

- ▶ Dunnett's for comparing  $\bar{\mu}_{1.}, \dots, \bar{\mu}_{a-1.}$  to a control mean  $\bar{\mu}_{a.}$ :

$$\bar{Y}_{i..} - \bar{Y}_{a..} \pm d_{a,ab(n-1),\alpha} \hat{\sigma} \sqrt{\frac{2}{bn}}, \quad i = 1, \dots, a - 1.$$

Still use  $\hat{\sigma} = \sqrt{\text{MS}_{\text{Error}}}$ .

Do likewise for making comparisons among  $\bar{\mu}_{.1}, \dots, \bar{\mu}_{.b}$  of Factor B.

## Serum glucose example from Kuehl (2000)

Two methods for measuring serum glucose level at three glucose levels.

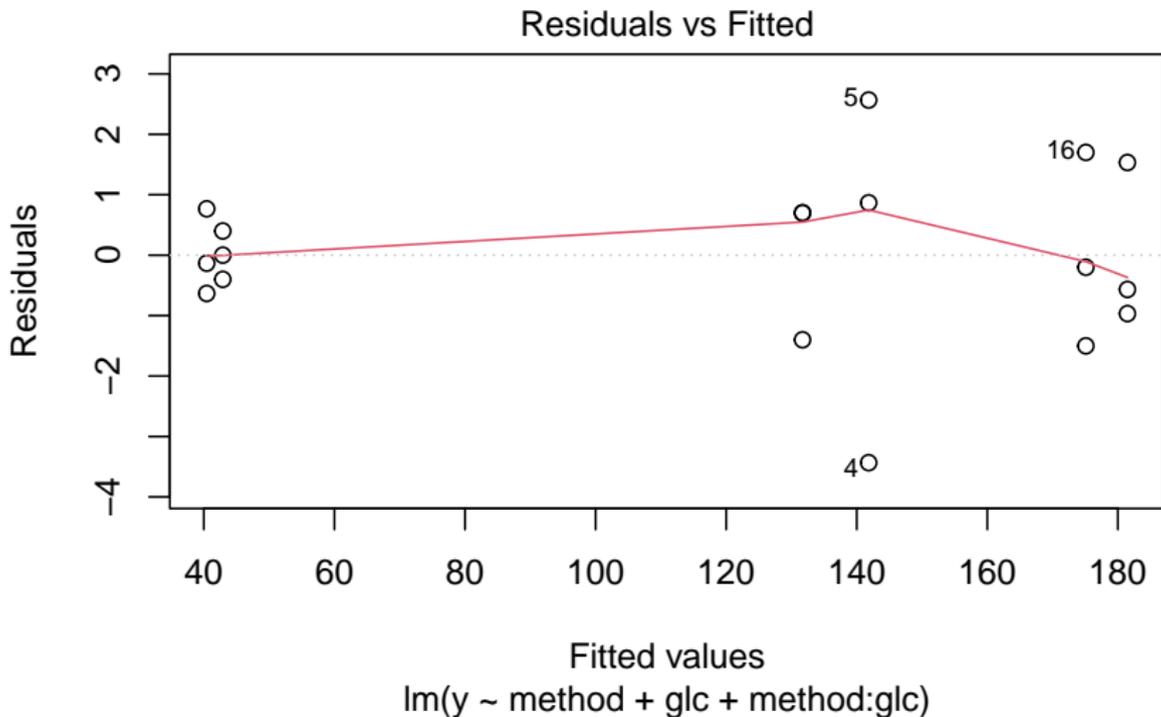
| <i>Glucose Level</i> | <i>Method 1</i> |          |          | <i>Method 2</i> |          |          |
|----------------------|-----------------|----------|----------|-----------------|----------|----------|
|                      | <i>1</i>        | <i>2</i> | <i>3</i> | <i>1</i>        | <i>2</i> | <i>3</i> |
|                      | 42.5            | 138.4    | 180.9    | 39.8            | 132.4    | 176.8    |
|                      | 43.3            | 144.4    | 180.5    | 40.3            | 132.4    | 173.6    |
|                      | 42.9            | 142.7    | 183.0    | 41.2            | 130.3    | 174.9    |

Source: Dr. J. Anderson, Beckman Instruments Inc.

```
y <- c(42.5,43.3,42.9,138.4,144.4,142.7,180.9,180.5,183.0,  
      39.8,40.3,41.2,132.4,132.4,130.3,176.8,173.6,174.9)  
method <- as.factor(c(1,1,1,1,1,1,1,1,1,2,2,2,2,2,2,2,2))  
glc <- as.factor(c(1,1,1,2,2,2,3,3,3,1,1,1,2,2,2,3,3,3))
```

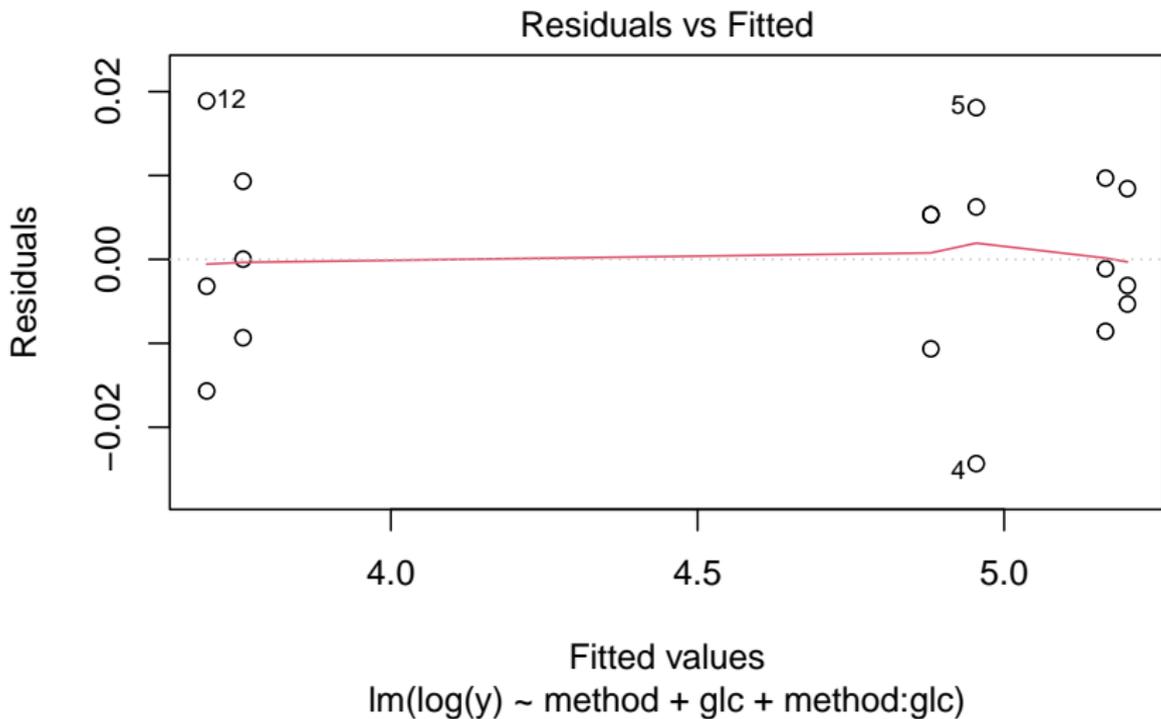
- ▶ Is there an interaction between the method and the glucose level?
- ▶ If not, can we describe the main effect of the method?

```
lm_glc <- lm(y ~ method + glc + method:glc)
plot(lm_glc, which = 1)
```



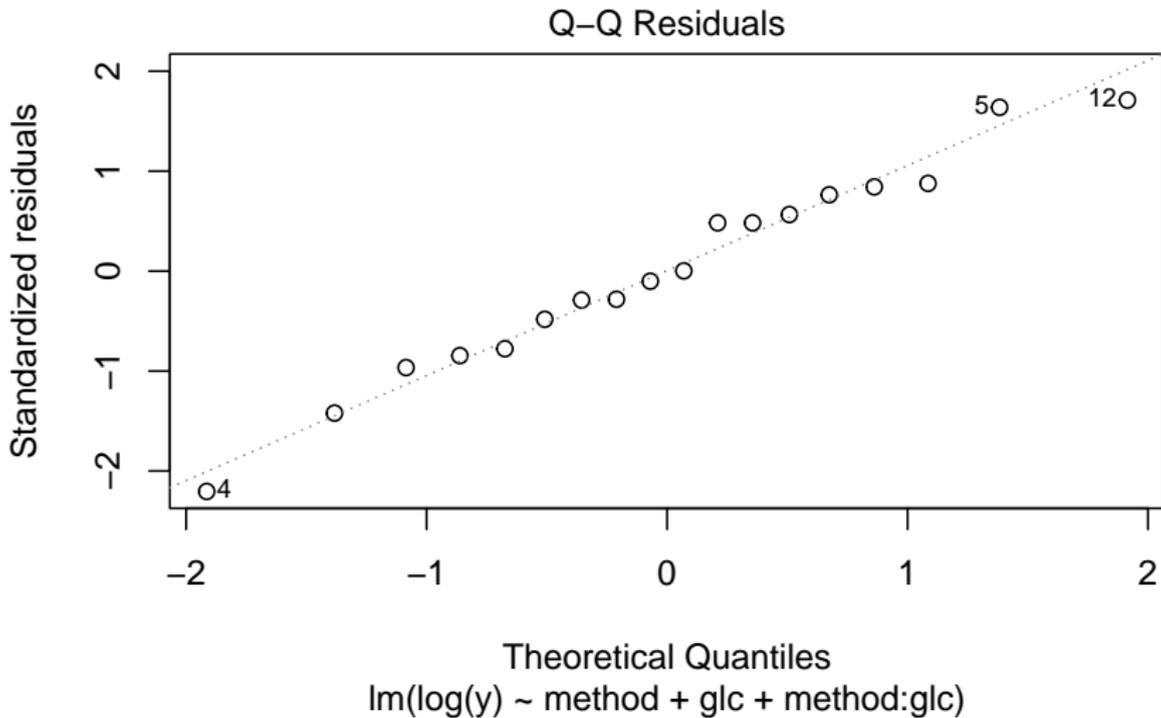
Variance appears smaller at lower glucose level. Try using  $\log(Y_{ijk})$ .

```
lm2_glc <- lm(log(y) ~ method + glc + method:glc)
plot(lm2_glc, which = 1)
```



This looks better.

```
plot(lm2_glc,which = 2)
```



Normality check looks okay.

```
anova(lm2_glc)
```

### Analysis of Variance Table

Response: log(y)

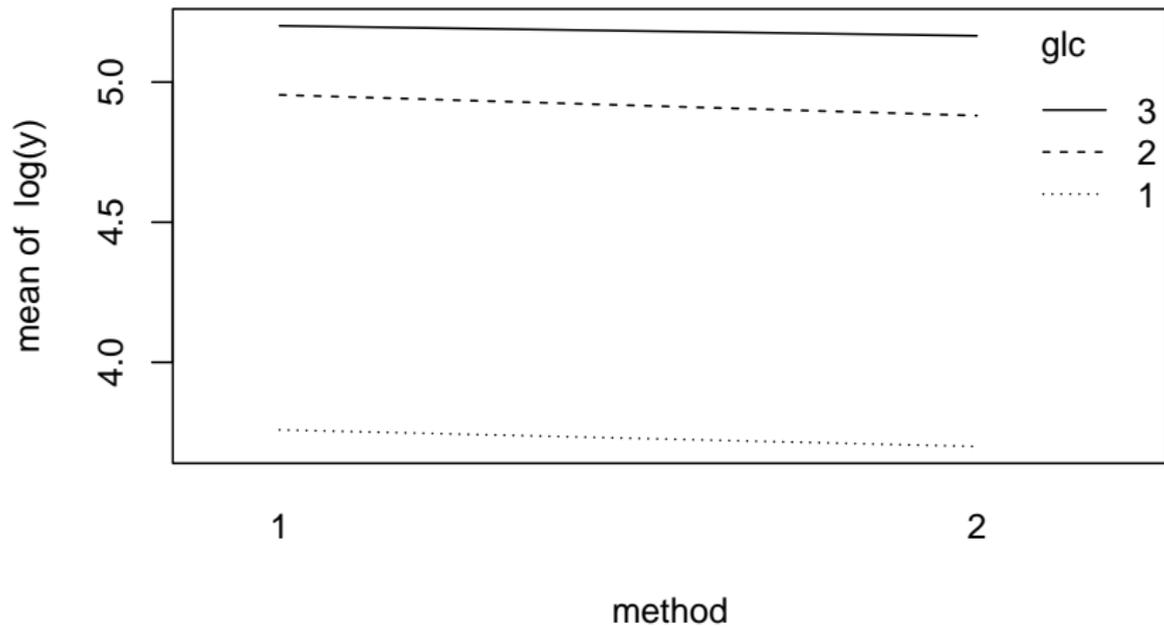
|            | Df | Sum Sq | Mean Sq | F value    | Pr(>F)    |     |
|------------|----|--------|---------|------------|-----------|-----|
| method     | 1  | 0.0143 | 0.0143  | 78.1091    | 1.337e-06 | *** |
| glc        | 2  | 7.1935 | 3.5967  | 19670.4837 | < 2.2e-16 | *** |
| method:glc | 2  | 0.0011 | 0.0006  | 3.0574     | 0.0845    | .   |
| Residuals  | 12 | 0.0022 | 0.0002  |            |           |     |

---

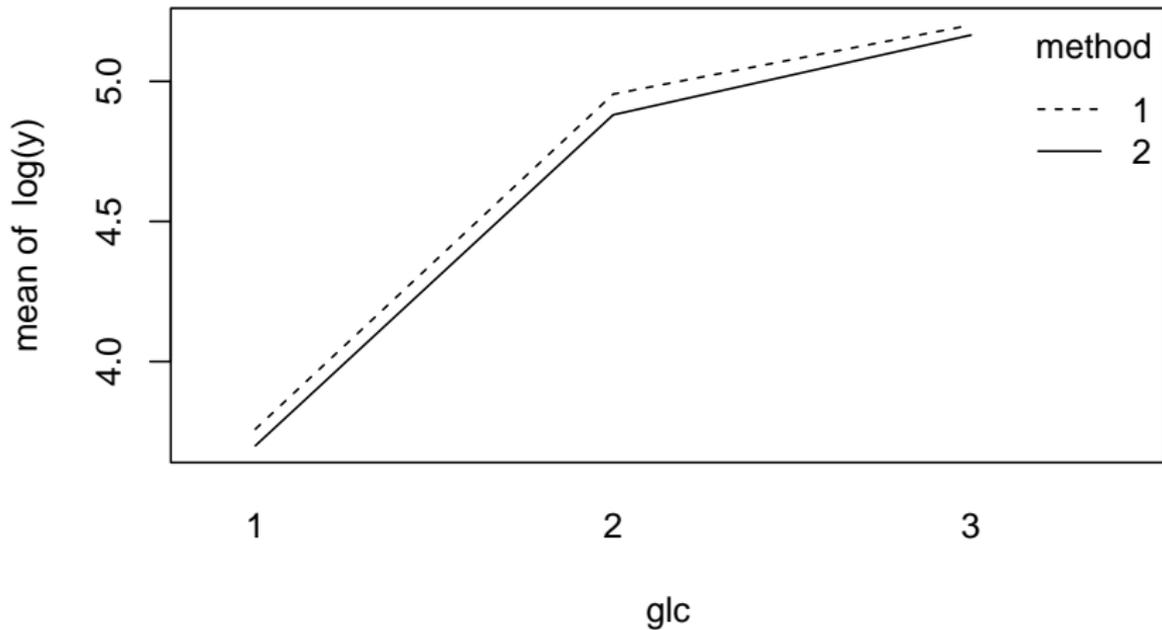
Signif. codes: 0 '\*\*\*' 0.001 '\*\*' 0.01 '\*' 0.05 '.' 0.1 ' ' 1

There is only weak evidence of interaction. Check interaction plot.

```
interaction.plot(method,glc,log(y))
```



```
interaction.plot(glc,method,log(y))
```



It appears safe to ignore the interaction and report on main effects.

Call the method Factor A; build a CI for  $\bar{\mu}_{1.} - \bar{\mu}_{2.}$  (just one comparison).

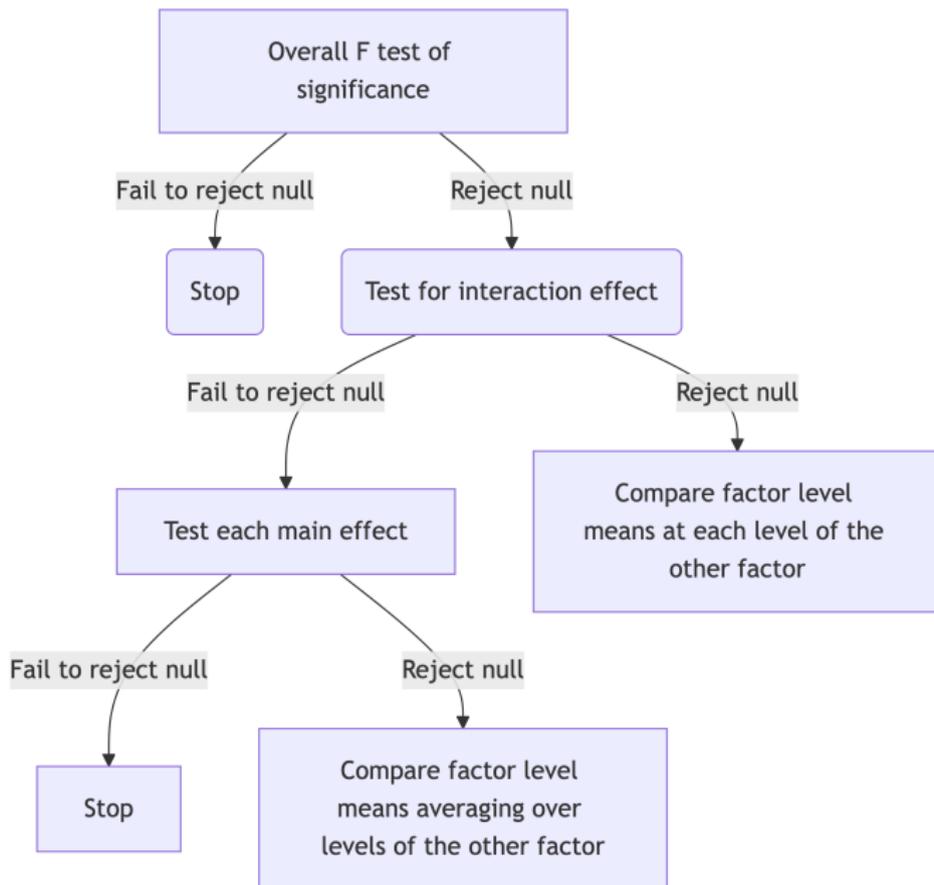
Since  $a = 2$ ,  $b = 3$ , and  $n = 3$ , use  $\bar{Y}_{1..} - \bar{Y}_{2..} \pm q_{2,2\cdot3(3-1),0.05} \hat{\sigma} \frac{1}{\sqrt{3 \cdot 3}}$ .

```
a <- 2
b <- 3
n <- 3
alpha <- 0.05
y1.. <- mean(log(y[method == 1])) # remember we are using log(y)
y2.. <- mean(log(y[method == 2]))
MSE <- sum(lm2_glc$residuals^2) / (a*b*(n-1))
me <- qtukey(1-alpha,a,a*b*(n-1)) * sqrt(MSE) / sqrt(n*b)
lo <- y1.. - y2.. - me
up <- y1.. - y2.. + me
c(lo,up)
```

```
[1] 0.04244809 0.07022545
```

Since  $a = 2$ ,  $q_{a,ab(n-1),\alpha} = \sqrt{2} \cdot t_{ab(n-1),\alpha/2}$ , so it is just a  $t$ -interval.

# Possible workflow for factorial experiments



# Three-way factorial experiment

Suppose we have factors A, B, and C with  $a$ ,  $b$ , and  $c$  levels, so that

$$Y_{ijkl} = \mu_{ijk} + \varepsilon_{ijkl}$$

for  $i = 1, \dots, a$ ,  $j = 1, \dots, b$ ,  $k = 1, \dots, c$ ,  $l = 1, \dots, n_{ijk}$ .

Treatment effects representation of the mean  $\mu_{ijk}$  is

$$\mu_{ijk} = \tau_i + \gamma_j + \eta_k + (\tau\gamma)_{ij} + (\tau\eta)_{ik} + (\gamma\eta)_{jk} + (\tau\gamma\eta)_{ijk}.$$

- ▶  $Y_{ijkl}$  is response for EU  $k$  under levels  $i, j, k$  of factors A, B, and C.
- ▶  $\mu$  represents a baseline or overall mean
- ▶ The  $\tau_i$ ,  $\gamma_j$ , and  $\eta_k$  are the main effects for Factors A, B, and C.
- ▶ The  $(\tau\gamma)_{ij}$ ,  $(\tau\eta)_{ik}$ , and  $(\gamma\eta)_{jk}$  are two-way interaction effects between the pairs of factors AB, AC, and BC.
- ▶ The  $(\tau\gamma\eta)_{ijk}$  are three-way interaction effects between A, B, and C.
- ▶ The  $\varepsilon_{ijkl}$  are  $\text{Normal}(0, \sigma^2)$  error terms.

## Shrimp data example

Four-week weight gain of shrimp in different aquarium settings:

| T   | D   | S   | Weight Gain (mg) | $\bar{y}_{ijk}$ | T  | D   | $\bar{y}_{ij..}$ | T  | S  | $\bar{y}_{i.k}$ | D   | S  | $\bar{y}_{.jk}$ |     |       |     |    |       |    |    |       |
|-----|-----|-----|------------------|-----------------|----|-----|------------------|----|----|-----------------|-----|----|-----------------|-----|-------|-----|----|-------|----|----|-------|
| 25° | 80  | 10% | 86,52,73         | 70.33           | 25 | 80  | 298.3            | 25 | 10 | 70.5            | 80  | 10 | 239.2           |     |       |     |    |       |    |    |       |
|     |     | 25% | 544,371,482      | 465.67          |    |     |                  |    |    |                 |     |    | 25              | 160 | 218.7 | 25  | 25 | 399.3 | 80 | 25 | 370.2 |
|     |     | 40% | 390,290,397      | 359             |    |     |                  |    |    |                 |     |    | 35              | 80  | 308.6 | 25  | 40 | 305.7 | 80 | 40 | 301.0 |
|     | 160 | 10% | 53,73,86         | 70.67           | 35 | 160 | 291.1            | 35 | 10 | 369.5           | 160 | 10 | 200.8           |     |       |     |    |       |    |    |       |
|     |     | 25% | 393,398,208      | 333             |    |     |                  |    |    |                 |     |    | 35              | 25  | 293.2 | 160 | 25 | 322.3 |    |    |       |
|     |     | 40% | 249,265,243      | 252.33          |    |     |                  |    |    |                 |     |    | 35              | 40  | 236.8 | 160 | 40 | 241.5 |    |    |       |
| 35° | 80  | 10% | 439,436,349      | 408             | 25 | 80  | 298.3            | 25 | 10 | 70.5            | 80  | 10 | 239.2           |     |       |     |    |       |    |    |       |
|     |     | 25% | 249,245,330      | 274.67          |    |     |                  |    |    |                 |     |    | 25              | 160 | 218.7 | 25  | 25 | 399.3 | 80 | 25 | 370.2 |
|     |     | 40% | 247,277,205      | 243             |    |     |                  |    |    |                 |     |    | 35              | 80  | 308.6 | 25  | 40 | 305.7 | 80 | 40 | 301.0 |
|     | 160 | 10% | 324,305,364      | 331             | 35 | 160 | 291.1            | 35 | 10 | 369.5           | 160 | 10 | 200.8           |     |       |     |    |       |    |    |       |
|     |     | 25% | 352,267,316      | 311.67          |    |     |                  |    |    |                 |     |    | 35              | 25  | 293.2 | 160 | 25 | 322.3 |    |    |       |
|     |     | 40% | 188,223,281      | 230.67          |    |     |                  |    |    |                 |     |    | 35              | 40  | 236.8 | 160 | 40 | 241.5 |    |    |       |

Temperature (Celcius), Density (# shrimp per 40 liters), and Salinity.  
Data from Kuehl (2000), image from Dr. Longnecker's notes.

```

wg <- c(86,52,73,544,371,482,390,290,397,
        53,73,86,393,398,208,249,265,243,
        439,436,349,249,245,330,247,277,205,
        324,305,364,352,267,316,188,223,281)
temp <- as.factor(c(rep(25,18),rep(35,18)))
dens <- as.factor(c(1,1) %x% c(rep(80,9),rep(160,9))) # kronecker product
salt <- as.factor(c(1,1,1,1) %x% c(rep(10,3),rep(25,3),rep(40,3)))
aggregate(wg ~ temp + dens + salt, FUN = mean)

```

|    | temp | dens | salt | wg        |
|----|------|------|------|-----------|
| 1  | 25   | 80   | 10   | 70.33333  |
| 2  | 35   | 80   | 10   | 408.00000 |
| 3  | 25   | 160  | 10   | 70.66667  |
| 4  | 35   | 160  | 10   | 331.00000 |
| 5  | 25   | 80   | 25   | 465.66667 |
| 6  | 35   | 80   | 25   | 274.66667 |
| 7  | 25   | 160  | 25   | 333.00000 |
| 8  | 35   | 160  | 25   | 311.66667 |
| 9  | 25   | 80   | 40   | 359.00000 |
| 10 | 35   | 80   | 40   | 243.00000 |
| 11 | 25   | 160  | 40   | 252.33333 |
| 12 | 35   | 160  | 40   | 230.66667 |

```
aggregate(wg ~ temp + dens, FUN = mean)
```

|   | temp | dens | wg       |
|---|------|------|----------|
| 1 | 25   | 80   | 298.3333 |
| 2 | 35   | 80   | 308.5556 |
| 3 | 25   | 160  | 218.6667 |
| 4 | 35   | 160  | 291.1111 |

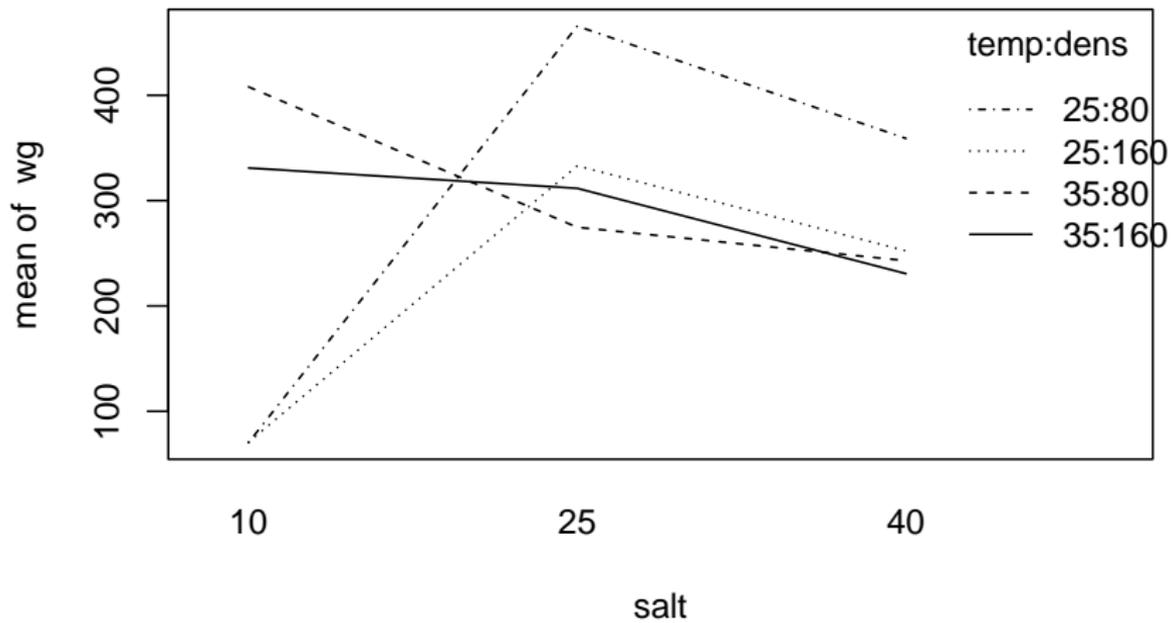
```
aggregate(wg ~ temp + salt, FUN = mean)
```

|   | temp | salt | wg       |
|---|------|------|----------|
| 1 | 25   | 10   | 70.5000  |
| 2 | 35   | 10   | 369.5000 |
| 3 | 25   | 25   | 399.3333 |
| 4 | 35   | 25   | 293.1667 |
| 5 | 25   | 40   | 305.6667 |
| 6 | 35   | 40   | 236.8333 |

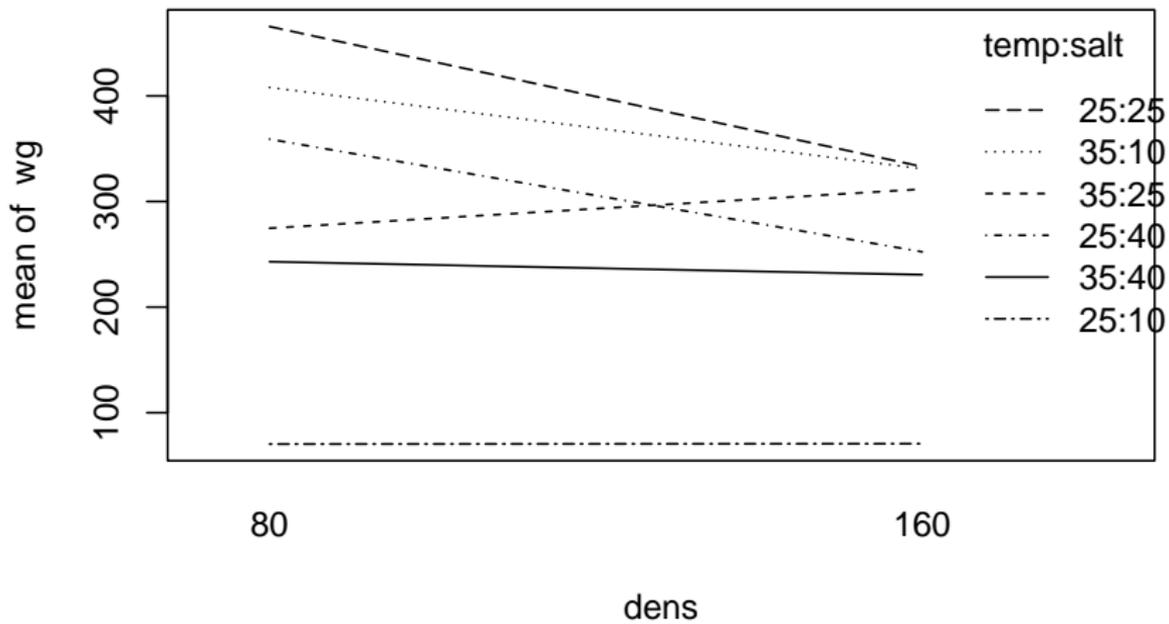
```
aggregate(wg ~ dens + salt, FUN = mean)
```

|   | dens | salt | wg       |
|---|------|------|----------|
| 1 | 80   | 10   | 239.1667 |
| 2 | 160  | 10   | 200.8333 |
| 3 | 80   | 25   | 370.1667 |
| 4 | 160  | 25   | 322.3333 |
| 5 | 80   | 40   | 301.0000 |
| 6 | 160  | 40   | 241.5000 |

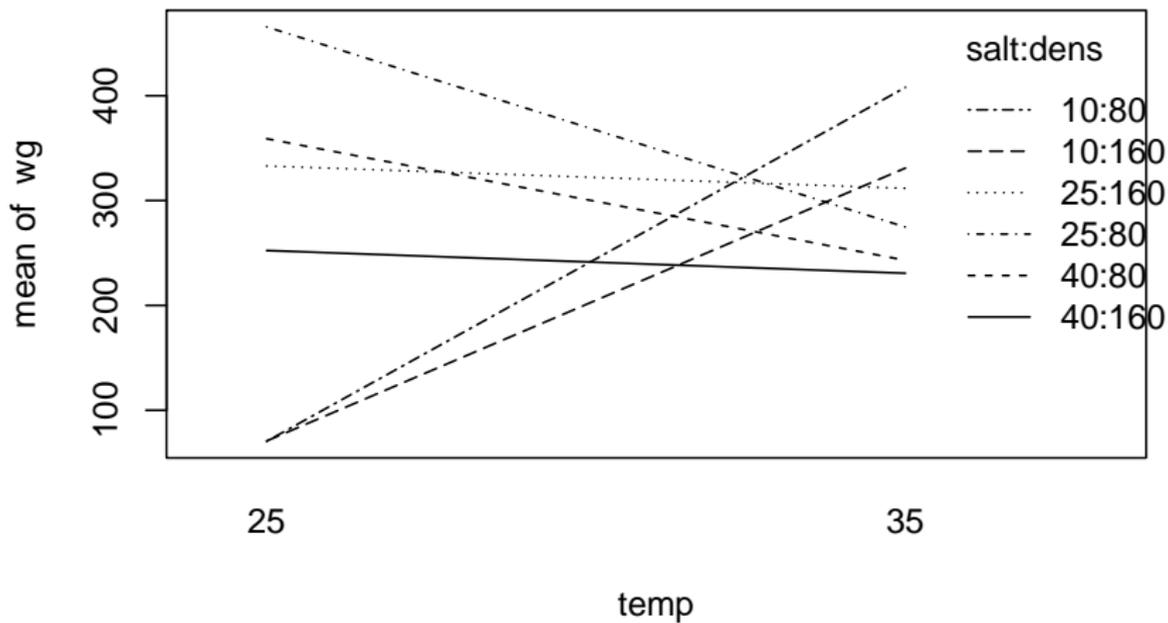
```
interaction.plot(salt,temp:dens,wg)
```



```
interaction.plot(dens,temp:salt,wg)
```



```
interaction.plot(temp,salt:dens,wg)
```



## ANOVA table for balanced three-way factorial design

| Source | Df                | SS                  | MS                  | F value                                  |
|--------|-------------------|---------------------|---------------------|--|
| A      | $a - 1$           | $SS_A$              | $MS_A$              | $F_A = MS_A / MS_{\text{Error}}$         |
| B      | $b - 1$           | $SS_B$              | $MS_B$              | $F_B = MS_B / MS_{\text{Error}}$         |
| C      | $c - 1$           | $SS_C$              | $MS_C$              | $F_C = MS_C / MS_{\text{Error}}$         |
| AB     | $(a - 1)(b - 1)$  | $SS_{AB}$           | $MS_{AB}$           | $F_{AB} = MS_{AB} / MS_{\text{Error}}$   |
| AC     | $(a - 1)(c - 1)$  | $SS_{AC}$           | $MS_{AC}$           | $F_{AC} = MS_{AC} / MS_{\text{Error}}$   |
| BC     | $(b - 1)(c - 1)$  | $SS_{BC}$           | $MS_{BC}$           | $F_{BC} = MS_{BC} / MS_{\text{Error}}$   |
| ABC    | $(a-1)(b-1)(c-1)$ | $SS_{ABC}$          | $MS_{ABC}$          | $F_{ABC} = MS_{ABC} / MS_{\text{Error}}$ |
| Error  | $abc(n - 1)$      | $SS_{\text{Error}}$ | $MS_{\text{Error}}$ |  |
| Total  | $abcn - 1$        | $SS_{\text{Tot}}$   |                     |  |

1. Reject  $H_0$ : no A main effect if  $F_A > F_{a-1, abc(n-1), \alpha}$ .
2. Reject  $H_0$ : no B main effect if  $F_B > F_{b-1, abc(n-1), \alpha}$ .
3. Reject  $H_0$ : no C main effect if  $F_C > F_{c-1, abc(n-1), \alpha}$ .
4. Reject  $H_0$ : no A and B interaction if  $F_{AB} > F_{(a-1)(b-1), abc(n-1), \alpha}$ .
5. Reject  $H_0$ : no A and C interaction if  $F_{AC} > F_{(a-1)(c-1), abc(n-1), \alpha}$ .
6. Reject  $H_0$ : no B and C interaction if  $F_{BC} > F_{(b-1)(c-1), abc(n-1), \alpha}$ .
7. Reject  $H_0$ : no A, B, and C interaction if  $F_{ABC} > F_{(a-1)(b-1)(c-1), abc(n-1), \alpha}$ .

## Shrimp data (cont)

```
lm_shrimp <- lm(wg ~ temp + dens + salt
                + temp:dens + temp:salt + dens:salt + temp:dens:salt)
anova(lm_shrimp)
```

### Analysis of Variance Table

Response: wg

|                | Df | Sum Sq | Mean Sq | F value | Pr(>F)    |     |
|----------------|----|--------|---------|---------|-----------|-----|
| temp           | 1  | 15376  | 15376   | 5.2952  | 0.03038   | *   |
| dens           | 1  | 21219  | 21219   | 7.3073  | 0.01242   | *   |
| salt           | 2  | 96762  | 48381   | 16.6615 | 2.901e-05 | *** |
| temp:dens      | 1  | 8711   | 8711    | 2.9999  | 0.09610   | .   |
| temp:salt      | 2  | 300855 | 150428  | 51.8041 | 1.959e-09 | *** |
| dens:salt      | 2  | 674    | 337     | 0.1161  | 0.89086   |     |
| temp:dens:salt | 2  | 24038  | 12019   | 4.1392  | 0.02855   | *   |
| Residuals      | 24 | 69691  | 2904    |         |           |     |

---

Signif. codes: 0 '\*\*\*' 0.001 '\*\*' 0.01 '\*' 0.05 '.' 0.1 ' ' 1

# References

Kuehl, R. O. 2000. *Design of Experiments: Statistical Principles of Research Design and Analysis*. Duxbury/Thomson Learning.

Mohr, Donna L, William J Wilson, and Rudolf J Freund. 2021. *Statistical Methods*. Academic Press.