STAT 516 Lec 12

Logistic regression

Karl Gregory

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Programming task data from Kutner et al. (2005)

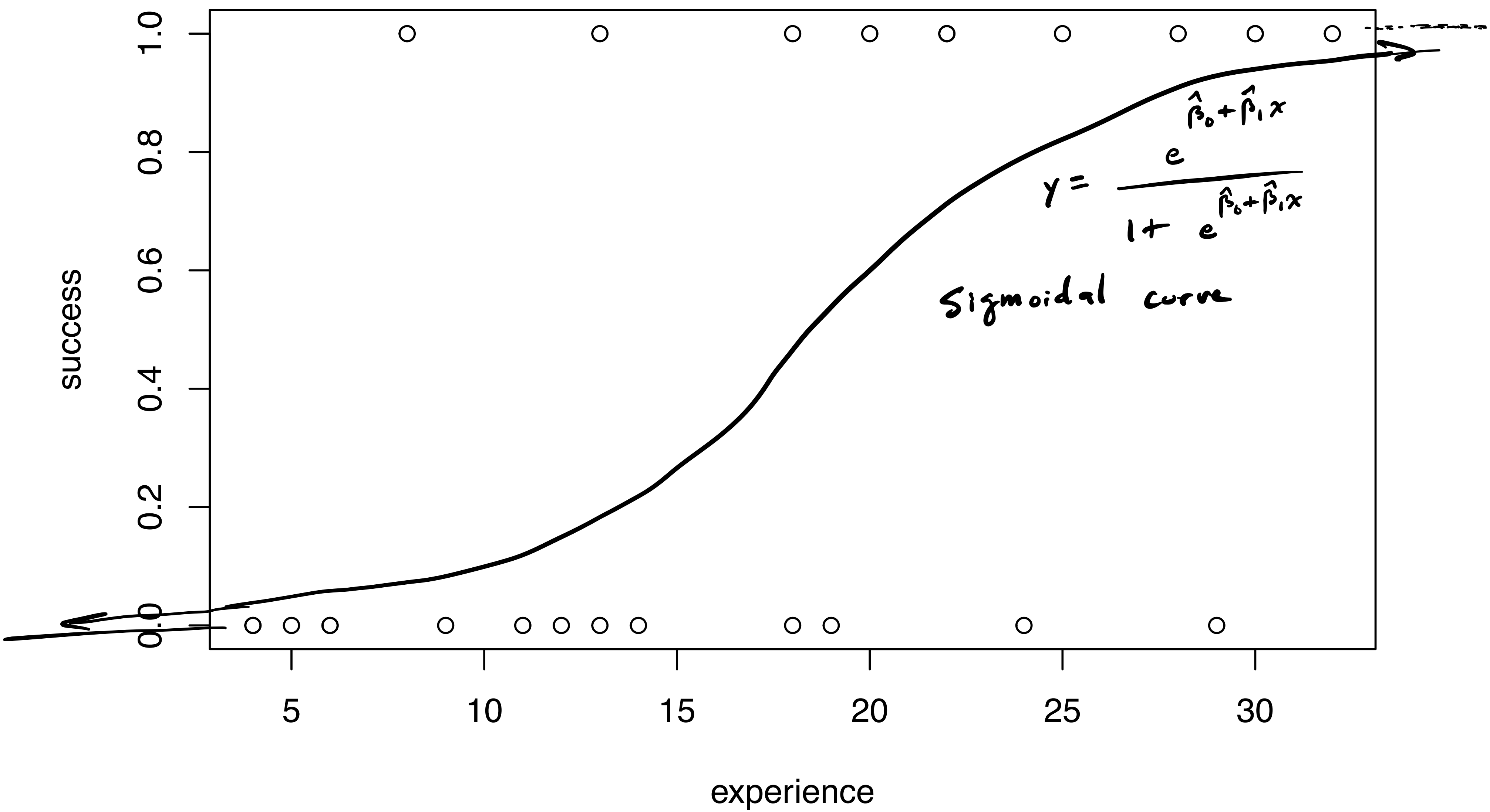
Twenty-five people succeeded or failed at a programming task.

Months of programming experience was recorded for each person.

```
experience <- c(14,29,6,25,18,4,18,12,22,6,30,11,30,5,20,13,9,32,24,13,19,4,28,22,8)
success <- c(0,0,0,1,1,0,0,0,1,0,1,0,1,0,1,0,0,1,0,1,0,0,1,1,1)
```

Can we predict probability of success based on experience?

```r
plot(success ~ experience)
```



$X \sim \text{Bernoulli}(p)$

$\Rightarrow P(X=x) = p^x (1-p)^{1-x}, \; x = 0, 1.$

Observe $(x_1, Y_1), \ldots, (x_n, Y_n)$, $x_i \in \mathbb{R}$, $Y_i \in \{0, 1\}$

Assume **logistic regression model**:

$$Y_i \sim \text{Bernoulli}(\pi_i), \quad \underbrace{\log\left(\frac{\pi_i}{1-\pi_i}\right)}_{\text{log-odds}} = \beta_0 + \beta_1 x_i$$

for $i = 1, \ldots, n$.

$\Leftrightarrow$ "Odds" $\dfrac{\pi_i}{1-\pi_i} = e^{\beta_0 + \beta_1 x_i}$

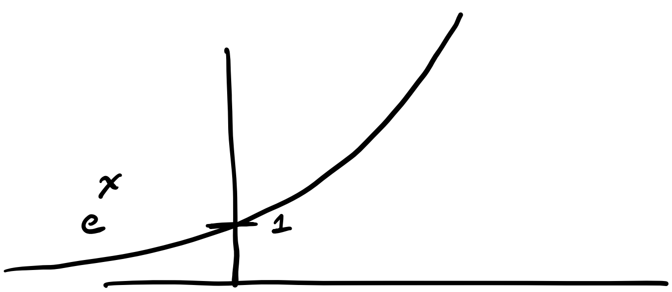
$\Leftrightarrow \dfrac{1-\pi_i}{\pi_i} = \dfrac{1}{e^{\beta_0 + \beta_1 x_i}}$

$\dfrac{1}{\pi_i} - 1 = \dfrac{1}{e^{\beta_0 + \beta_1 x_i}}$

$\dfrac{1}{\pi_i} = 1 + \dfrac{1}{e^{\beta_0 + \beta_1 x_i}}$

$\dfrac{1}{\pi_i} = \dfrac{e^{\beta_0 + \beta_1 x_i} + 1}{e^{\beta_0 + \beta_1 x_i}}$

$\pi_i = \dfrac{e^{\beta_0 + \beta_1 x_i}}{1 + e^{\beta_0 + \beta_1 x_i}} \in (0, 1)$



$e^{\beta_0 + \beta_1 x_i} > 0$

$\frac{\pi}{1-\pi}$ Odds of "success" for Y_i.

$$\text{Odds} = \frac{P(\text{"success"})}{1 - P(\text{"success"})}$$

Suppose $\pi = 1/2$. Then odds $\frac{\pi}{1-\pi} = \frac{1/2}{1-1/2} = 1$.

$\pi = 1/3$. Then odds $= \frac{1/3}{1-1/3} = \frac{1/3}{2/3} = \frac{1}{2}$.

$\pi = 2/3$. Odds $= \frac{2/3}{1-2/3} = \frac{2/3}{1/3} = 2$

$\pi = \frac{3}{5}$. Odds $\frac{3/5}{1-3/5} = \frac{3/5}{2/5} = \frac{3}{2}$

Logistic regression model

Assume

$$Y_i \sim \text{Bernoulli}(\pi_i), \quad \log\left(\frac{\pi_i}{1-\pi_i}\right) = \beta_0 + \beta_1 x_i,$$

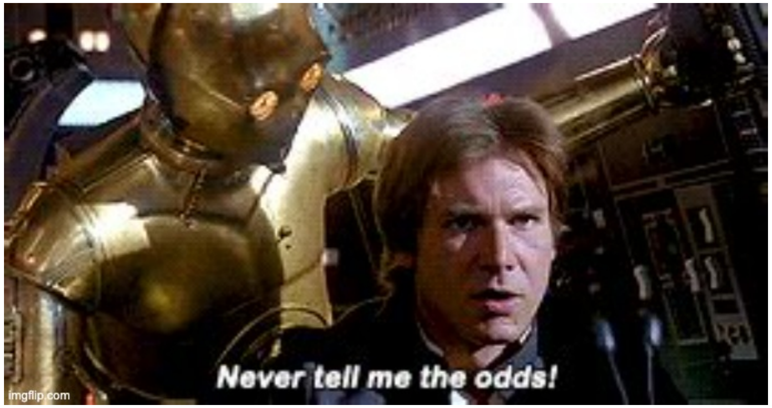
for $i = 1, \dots, n$, where

Y1,..., Yn are independent.

- Y_i is the response for observation i.
- x_i is the value of a predictor/covariate/explanatory variable for obs i.
- π_i is the probability of "success" for observation i.
- β_0 and β_1 are slope and intercept parameters.
- $\pi_i/(1-\pi_i)$ is the odds of "success" for obs i.
- $\log(\pi_i/(1-\pi_i))$ is the log-odds for obs i.

Logistic regression assumes the log-odds are linear in the predictor.

Odds



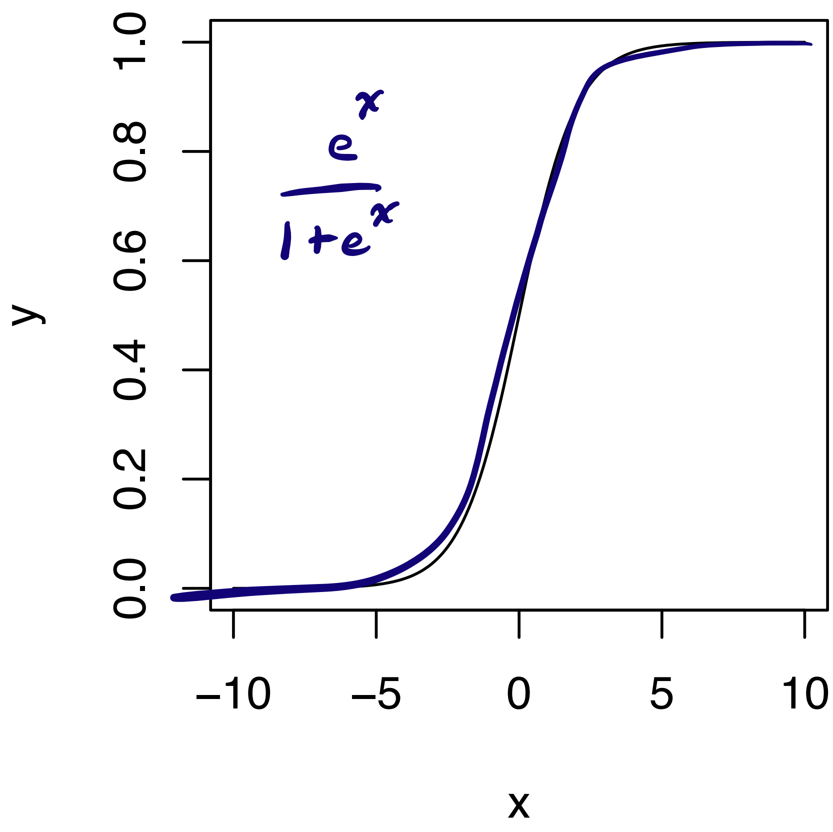
- Let π be the probability of success.
- Then $\pi/(1-\pi)$ is called the odds in favor of success.
  a. If $\pi = 1/2$ then $\pi/(1-\pi) = 1$. "One-to-one" odds of success.
  b. If $\pi = 2/3$ then $\pi/(1-\pi) = 2$. Success 2x more likely than failure.
  c. If $\pi = 1/4$ then $\pi/(1-\pi) = 1/3$. Failure 3x more likely than success.

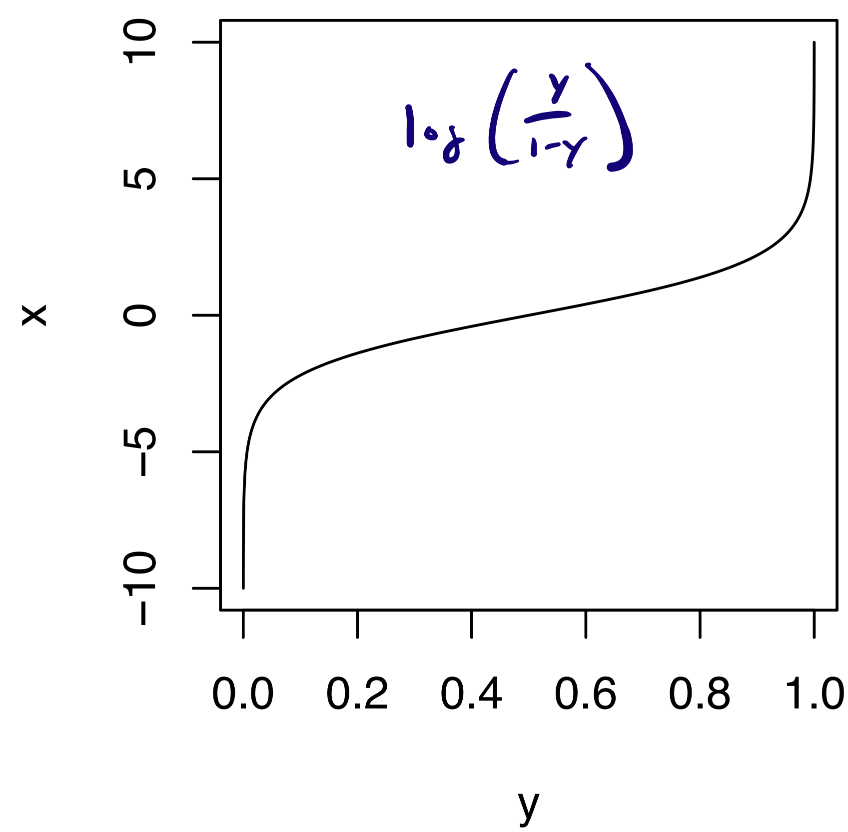
The logit and logistic transformations

- The transformation $y = \frac{e^x}{1+e^x}$ is called the logistic transformation.
- Its inverse $x = \log(\frac{y}{1-y})$ is called the logit transformation.
- We have

$$\log\left(\frac{\pi_i}{1-\pi_i}\right) = \beta_0 + \beta_1 x_i \iff \pi_i = \frac{e^{\beta_0+\beta_1 x_i}}{1+e^{\beta_0+\beta_1 x_i}}$$

```
x <- seq(-10,10,length=200)
y <- exp(x) / (1 + exp(x))
par(mfrow= c(1,2))
plot(y~x,type = "l")
plot(x~y,type = "l")
```





Goals in logistic regression

$Y_i \sim \text{Bernoulli}(\pi_i)$, $\log\left(\frac{\pi_i}{1-\pi_i}\right) = \beta_0 + \beta_1 x_i$

$\mathbb{E} Y_i = \pi_i$

$\text{Var } Y_i = \pi_i(1-\pi_i)$

$$\hat{\pi}_i = \frac{e^{\hat{\beta}_0 + \hat{\beta}_1 x_i}}{1 + e^{\hat{\beta}_0 + \hat{\beta}_1 x_i}}$$

1. Estimate β_0 and β_1.
2. Obtain fitted probabilities $\hat{\pi}_1, \ldots, \hat{\pi}_n$.
3. Build CI for β_1 and test H_0: $\beta_1 = 0$.
4. Give interpretations of the estimated regression coefficients.
5. Check goodness of fit of the logistic regression model.
6. Add additional covariates...

$X \sim \text{Bernoulli}(p)$

$P(X=x) = p^x(1-p)^{1-x} \quad x = 0, 1$

$\mathbb{E} X = p$

$\text{Var } X = p(1-p)$

How to estimate β_0, β_1 ?

Use Maximum Likelihood.

Say I observed $Y_1^{obs}, \dots, Y_n^{obs}$.

A, B indep.

$P(A \cap B) = P(A) P(B)$

Write down

$$P(Y_1 = Y_1^{obs}, Y_2 = Y_2^{obs}, \dots, Y_n = Y_n^{obs})$$

$$= P(Y_1 = Y_1^{obs}) * \dots * P(Y_n = Y_n^{obs})$$

$$= \pi_1^{Y_1^{obs}} (1-\pi_1)^{1-Y_1^{obs}} * \dots * \pi_n^{Y_n^{obs}} (1-\pi_n)^{1-Y_n^{obs}}$$

$$= \prod_{i=1}^{n} \pi_i^{Y_i^{obs}} (1-\pi_i)^{1-Y_i^{obs}}$$

($\prod$: product)

$$= \mathcal{L}(\beta_0, \beta_1, Y_1^{obs}, \dots, Y_n^{obs})$$

↑ "Likelihood function"

Maximum likelihood estimation in logistic regression

MLE

- We do not use least-squares to estimate β_0 and β_1.
- Instead we use maximum likelihood estimators (MLEs).
- The MLEs are the parameter values giving the observed data the highest possible probability.
- Intercept b_0 and slope b_1 give to the observed data the probability

$$\mathcal{L}_n(b_0, b_1) = \prod_{i=1}^{n} [\pi_i(b_0, b_1)]^{Y_i} [1 - \pi_i(b_0, b_1)]^{1-Y_i}$$

with $\pi_i(b_0, b_1) = \dfrac{e^{b_0 + b_1 x_i}}{1 + e^{b_0 + b_1 x_i}}$ for $i = 1, \dots, n$.

- The MLEs $\hat{\beta}_0, \hat{\beta}_1$ are the values of b_0, b_1 that maximize $\mathcal{L}_n(b_0, b_1)$.
- $\mathcal{L}_n(b_0, b_1)$ is called the likelihood function.

Computing the MLEs in logistic regression

SLR: $Q_n(b_0, b_1) = \sum_{i=1}^{n} (Y_i - (b_0 + b_1 x_i))^2$ Least-squares.

- There is no "closed-form" expression for $\hat{\beta}_0$ and $\hat{\beta}_1$.
- One must find their values numerically, that is with an algorithm.
- More convenient to work with $\log \mathcal{L}_n(b_0, b_1)$, which is given by

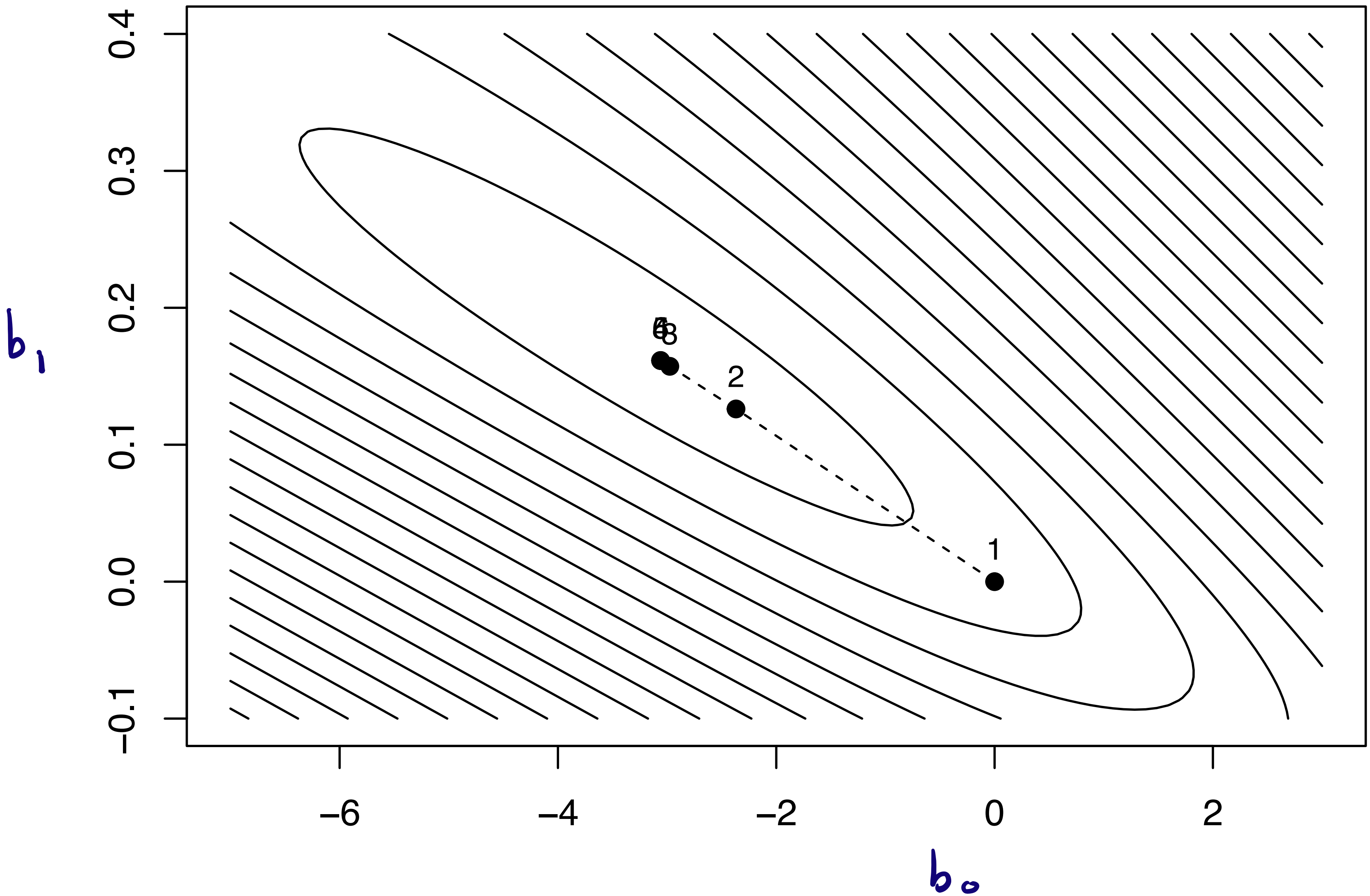
$$\ell_n(b_0, b_1) = \sum_{i=1}^{n} [Y_i(b_0 + b_1 x_i) - \log(1 + e^{b_0 + b_1 x_i})].$$

maximize

- Newton's method is one way to find the maximizers of $\ell_n(b_0, b_1)$.

Programming task data (cont)

Newton-Raphson algorithm for computing $\hat{\beta}_0$ and $\hat{\beta}_1$.



Generalized linear models (GLM)

$Y_i \sim \text{Bernoulli}(\pi_i)$, $\quad \log\left(\frac{\pi_i}{1-\pi_i}\right) = \beta_0 + \beta_1 x$

$\mathbb{E}\, Y_i = \pi_i$ $\quad$ non-linear $\quad$ linear

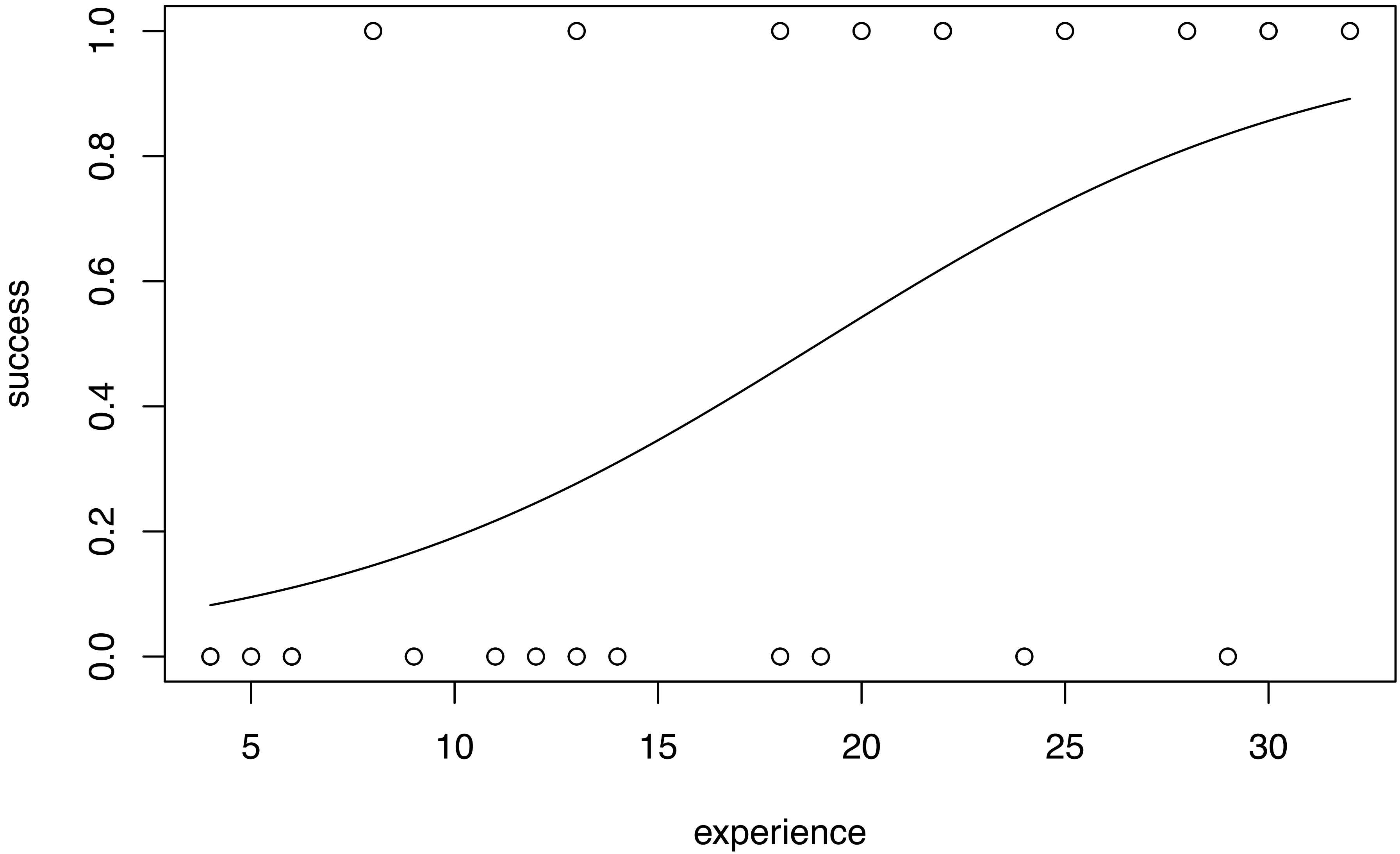
- The logistic regression model is in a class of models called GLMs.
- GLM stands for generalized linear model.
- Poisson regression, binomial response regression, i.a. are GLMs too.
- Use `glm()` function in R to obtain $\hat{\beta}_0$ and $\hat{\beta}_1$.

Use `glm()` function with the option `family = "binomial"`.

```r
glm_out <- glm(success ~ experience, family = "binomial")
summary(glm_out)
```

```
Call:
glm(formula = success ~ experience, family = "binomial")

Coefficients:
            Estimate Std. Error z value Pr(>|z|)
(Intercept) -3.05970    1.25935  -2.430   0.0151 *
experience   0.16149    0.06498   2.485   0.0129 *
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

(Dispersion parameter for binomial family taken to be 1)

    Null deviance: 34.296  on 24  degrees of freedom
Residual deviance: 25.425  on 23  degrees of freedom
AIC: 29.425

Number of Fisher Scoring iterations: 4
```

```r
x <- seq(min(experience),max(experience),length = 200)
pihat_x <- 1/(1 + exp( -(coef(glm_out)[1] + coef(glm_out)[2]*x)))
plot(success ~ experience); lines(pihat_x~x)
```



Fitted probabilities

- Define the fitted probabilities as

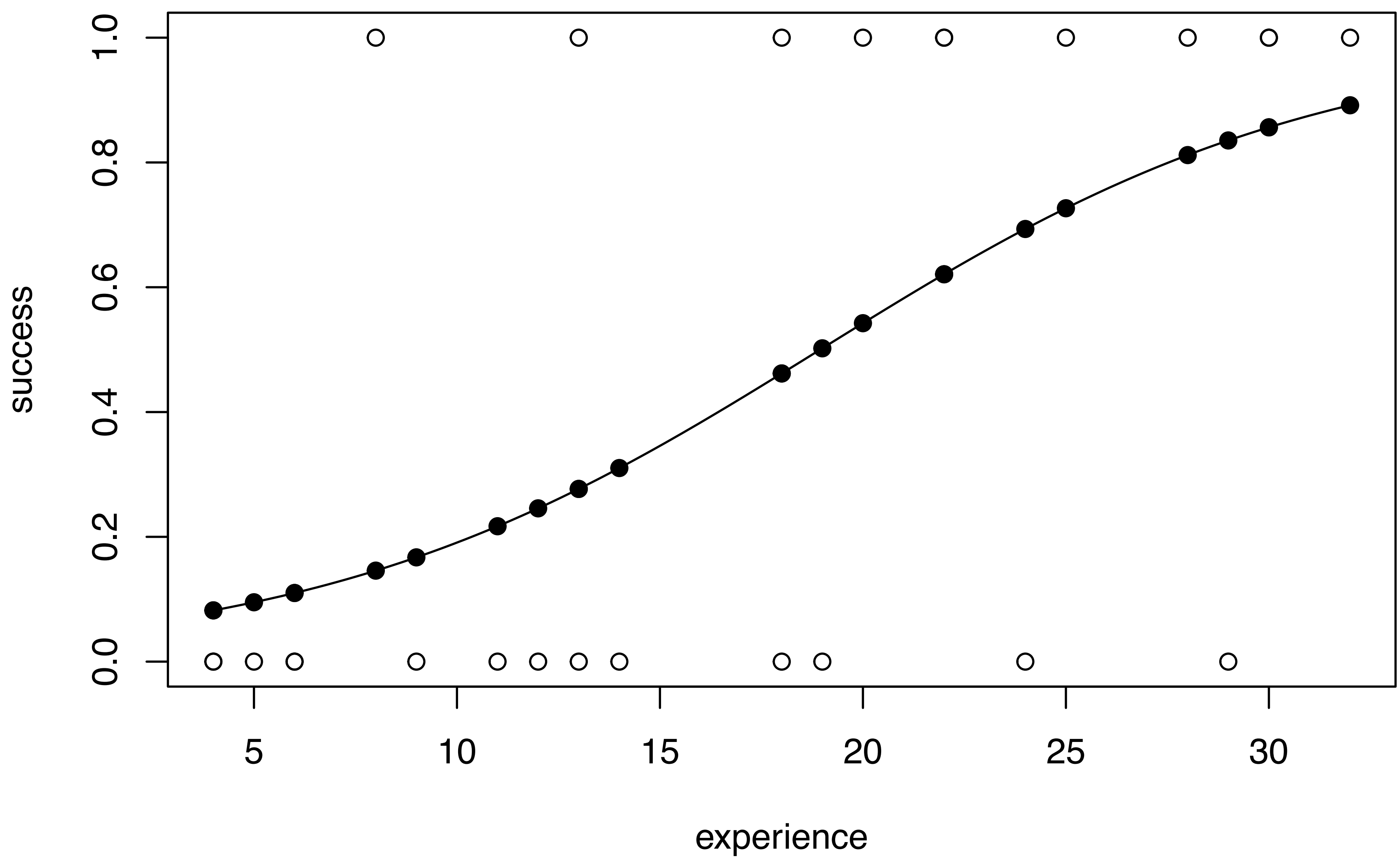
$$\hat{\pi}_i = \frac{e^{\hat{\beta}_0 + \hat{\beta}_1 x_i}}{1 + e^{\hat{\beta}_0 + \hat{\beta}_1 x_i}} \quad \text{for} \quad i = 1, \dots, n.$$

- For any value x_{new}, we estimate the probability of "success" as

$$\hat{\pi}_{\text{new}} = \frac{e^{\hat{\beta}_0 + \hat{\beta}_1 x_{\text{new}}}}{1 + e^{\hat{\beta}_0 + \hat{\beta}_1 x_{\text{new}}}}.$$

Programming task data (cont)

```
plot(success ~ experience); lines(pihat_x~x)
points(glm_out$fitted.values~experience,pch = 19)
```

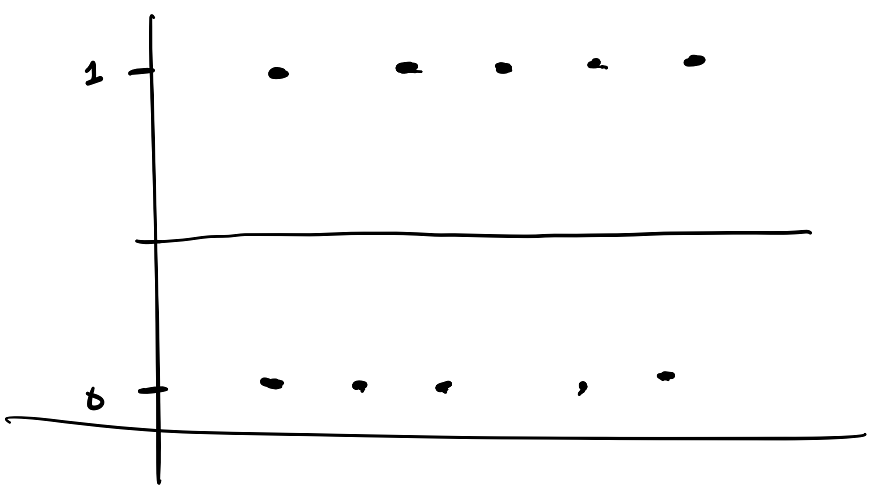


$H_0: \beta_1 = 0$ vs $H_1: \beta_1 \neq 0$.

For large n $\quad \hat{\beta}_1 \overset{\text{approx}}{\sim} N(\beta_1, \bigcirc)$

↑ complicated variance.

$$\Rightarrow \quad \frac{\hat{\beta}_1 - \beta_1}{\sqrt{\bigcirc}} \overset{\text{approx}}{\sim} N(0,1)$$



$$\frac{e^{\beta_0 + \beta_1 x}}{1 + e^{\beta_0 + \beta_1 x}}$$

$\beta_1 = 0$

Asymptotic distribution of slope estimator and CI

- For large enough n, $\hat{\beta}_1$ is approximately Normal, such that

$$\frac{\hat{\beta}_1 - \beta_1}{\widehat{\text{se}}\{\hat{\beta}_1\}} \overset{\text{approx}}{\sim} \text{Normal}(0, 1),$$

estimated standard error

where, setting $\hat{w}_i = \hat{\pi}_i(1 - \hat{\pi}_i)$ for $i = 1, \ldots, n$, we may write

$$\widehat{\text{se}}\{\hat{\beta}_1\} = \left[\sum_{i=1}^n \hat{w}_i x_i^2 - (\sum_{i=1}^n \hat{w}_i)^{-1}(\sum_{i=1}^n \hat{w}_i x_i)^2\right]^{-\frac{1}{2}}.$$

- We can make an approximate $(1 - \alpha)100\%$ CI for β_1 as

$$\hat{\beta}_1 \pm z_{\alpha/2}\, \widehat{\text{se}}\{\hat{\beta}_1\}.$$

Programming task data (cont)

```
# Confidence interval for beta1
pihat <- glm_out$fitted.values
b1hat <- coef(glm_out)[2]
w <- pihat*(1-pihat)
se <- sqrt(1/(sum(w*experience^2) - sum(w*experience)^2/sum(w)))
lo <- b1hat - 1.96 * se
up <- b1hat + 1.96 * se
c(lo,up)
```

```
experience experience
0.03412491 0.28884692
```

```
# CIs for both beta0 and beta1 automatically from glm_out
confint.default(glm_out)
```

```
                  2.5 %     97.5 %
(Intercept) -5.52797622 -0.5914155
experience   0.03412744  0.2888444
```

Testing whether the slope coefficient is zero

To test H_0: $\beta_1 = 0$ versus H_1: $\beta_1 \neq 0$, do:

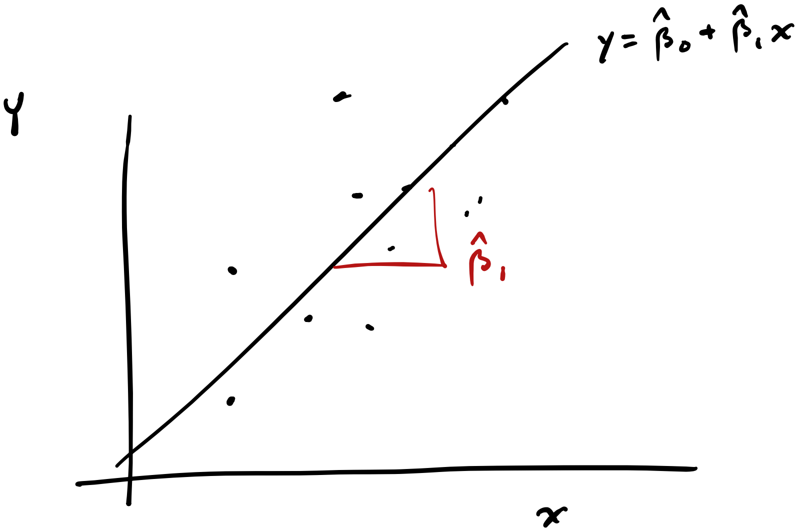
1. Compute $Z_{\text{test}} = \frac{\hat{\beta}_1}{\widehat{\text{se}}\{\hat{\beta}_1\}}$.
2. Reject H_0 at α if $|Z_{\text{test}}| > z_{\alpha/2}$.) OR just check if C.I. contains 0.
3. Obtain p value as $2(1 - P(Z > |Z_{\text{test}}|))$, $Z \sim$ Normal$(0, 1)$.

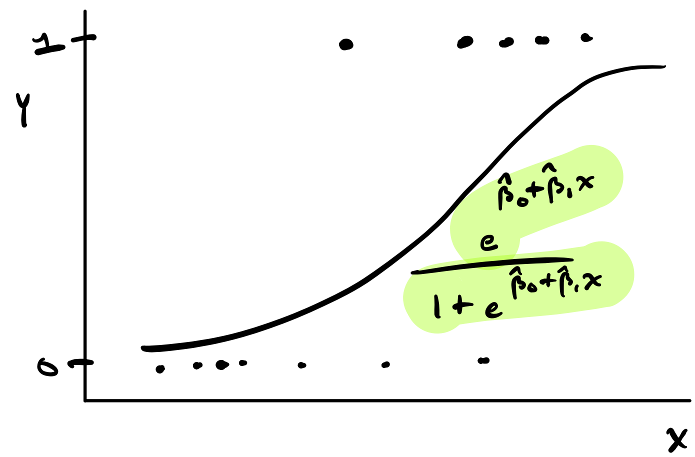
The `summary()` function on the `glm()` output prints this p value.

$$Y_i \sim \text{Bernoulli}(\pi_i)$$

$$\underbrace{\log\left(\frac{\pi_i}{1-\pi_i}\right)}_{\text{log-odds}} = \beta_0 + \beta_1 x_i$$

$i = 1, \ldots, n.$





How to interpret β_1

set $\quad \pi_0 = \dfrac{e^{\beta_0+\beta_1 x_0}}{1+e^{\beta_0+\beta_1 x_0}}$ $\qquad x_0$: "initial" value for x

$$\pi_1 = \frac{e^{\beta_0+\beta_1(x_0+1)}}{1+e^{\beta_0+\beta_1(x_0+1)}}.$$

Now write

$$\log\left(\frac{\pi_0}{1-\pi_0}\right) = \beta_0 + \beta_1 x_0$$

$$\log\left(\frac{\pi_1}{1-\pi_1}\right) = \beta_0 + \beta_1 (x_0+1)$$

$$\log\left(\frac{\pi_1}{1-\pi_1}\right) - \log\left(\frac{\pi_0}{1-\pi_0}\right) = \beta_0 + \beta_1(x_0+1) - (\beta_0 + \beta_1 x_0)$$

$$= \beta_1$$

↑

Amount by which the log-odds changes due to a 1-unit increase in x.

$$\log a - \log b = \log\left(\frac{a}{b}\right)$$

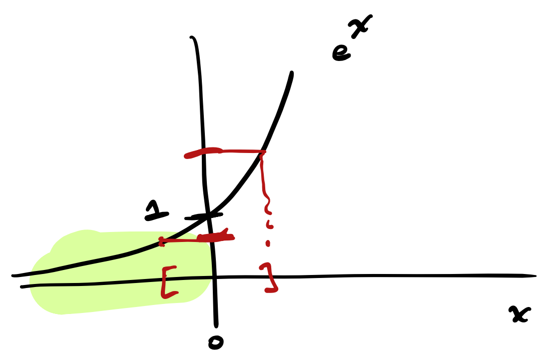
$\iff$

$$\log\left(\frac{\frac{\pi_1}{1-\pi_1}}{\frac{\pi_0}{1-\pi_0}}\right) = \beta_1$$

$\frac{\text{odds}_0}{\text{odds}_1}$ "Odds ratio"

$\iff$

$$\frac{\frac{\pi_1}{1-\pi_1}}{\frac{\pi_0}{1-\pi_0}} = e^{\beta_1}$$



$\iff$

$$\frac{\pi_1}{1-\pi_1} = e^{\beta_1} \frac{\pi_0}{1-\pi_0}$$

programming: $\hat{\beta}_1 = 0.1615$, $e^{\hat{\beta}_1} = 1.175$

Each additional month of experience is estimated to increase the odds of success by (i) a factor of 1.175.

Odds ratios

(ii) 17.5%

- Let π_0 and π_1 be success probs under an initial and an altered condition, respectively.
- Then we call the ratio
$$\frac{\pi_1/(1-\pi_1)}{\pi_0/(1-\pi_0)}$$
the odds ratio associated with the change from the initial to the altered condition.
- The odds ratio is the factor by which the odds are multiplied when the initial condition is changed to the altered condition.

Interpreting the logistic regression parameters

- Let π_0 and π_1 be the "success" probabilities at x_0 and $x_0 + 1$.
- Then we have the two equations
  1. $\log\left(\frac{\pi_0}{1-\pi_0}\right) = \beta_0 + \beta_1 x_0$
  2. $\log\left(\frac{\pi_1}{1-\pi_1}\right) = \beta_0 + \beta_1 (x_0 + 1)$
- Subtracting the first equation from the second gives

$$\beta_1 = \log\left(\frac{\pi_1}{1-\pi_1}\right) - \log\left(\frac{\pi_0}{1-\pi_0}\right) = \log\left(\frac{\pi_1/(1-\pi_1)}{\pi_0/(1-\pi_0)}\right).$$

- The quantity $\dfrac{\pi_1/(1-\pi_1)}{\pi_0/(1-\pi_0)}$ is called an odds ratio.
- So β_1 is log of the odds ratio associated with a unit increase in x.

Odds ratio from a unit increase in x

C.I. for β_1: $\hat{\beta}_1 \pm z_{\alpha/2} \widehat{se}\{\hat{\beta}_1\}$

- From the previous slide, we have

$$e^{\beta_1} = \frac{\pi_1/(1-\pi_1)}{\pi_0/(1-\pi_0)}.$$

- Can build a CI for e^{β_1} by exponentiating the CI for β_1.
- Gives CI for e^{β_1} as $[e^{\hat{\beta}_1 - z_{\alpha/2}\widehat{\text{se}}\{\hat{\beta}_1\}}, e^{\hat{\beta}_1 + z_{\alpha/2}\widehat{\text{se}}\{\hat{\beta}_1\}}]$.
- A unit increase in x multiplies the odds of success by the factor e^{β_1}.
- What if the CI for e^{β_1} contains 1?

C.I. for e^{β_1} contains 1 $\iff$ (if and only if) C.I. for β_1 contains zero

Programming task data (cont)

```
exp(confint.default(glm_out,parm = "experience"))
```

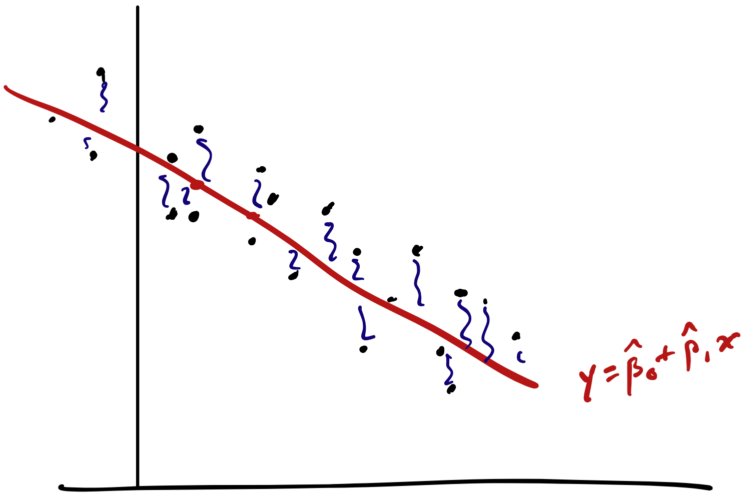
```
              2.5 %   97.5 %
experience 1.034716 1.334884
```

Each additional month of experience increases the odds of completing the programming task by a factor of 1.035 to 1.335, with 95% confidence.

SLR

$$Y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$$

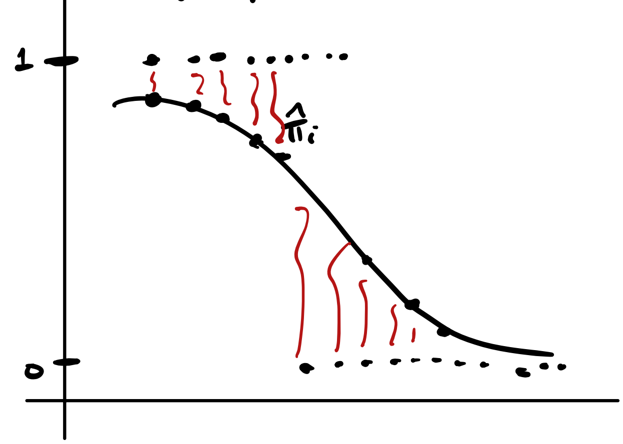
$$\varepsilon_i \sim N(0, \sigma^2)$$



Logistic R

$$Y_i \sim \text{Bernoulli}(\pi_i)$$

$$\log\left(\frac{\pi_i}{1-\pi_i}\right) = \beta_0 + \beta_1 x_i$$



Residuals for logistic regression

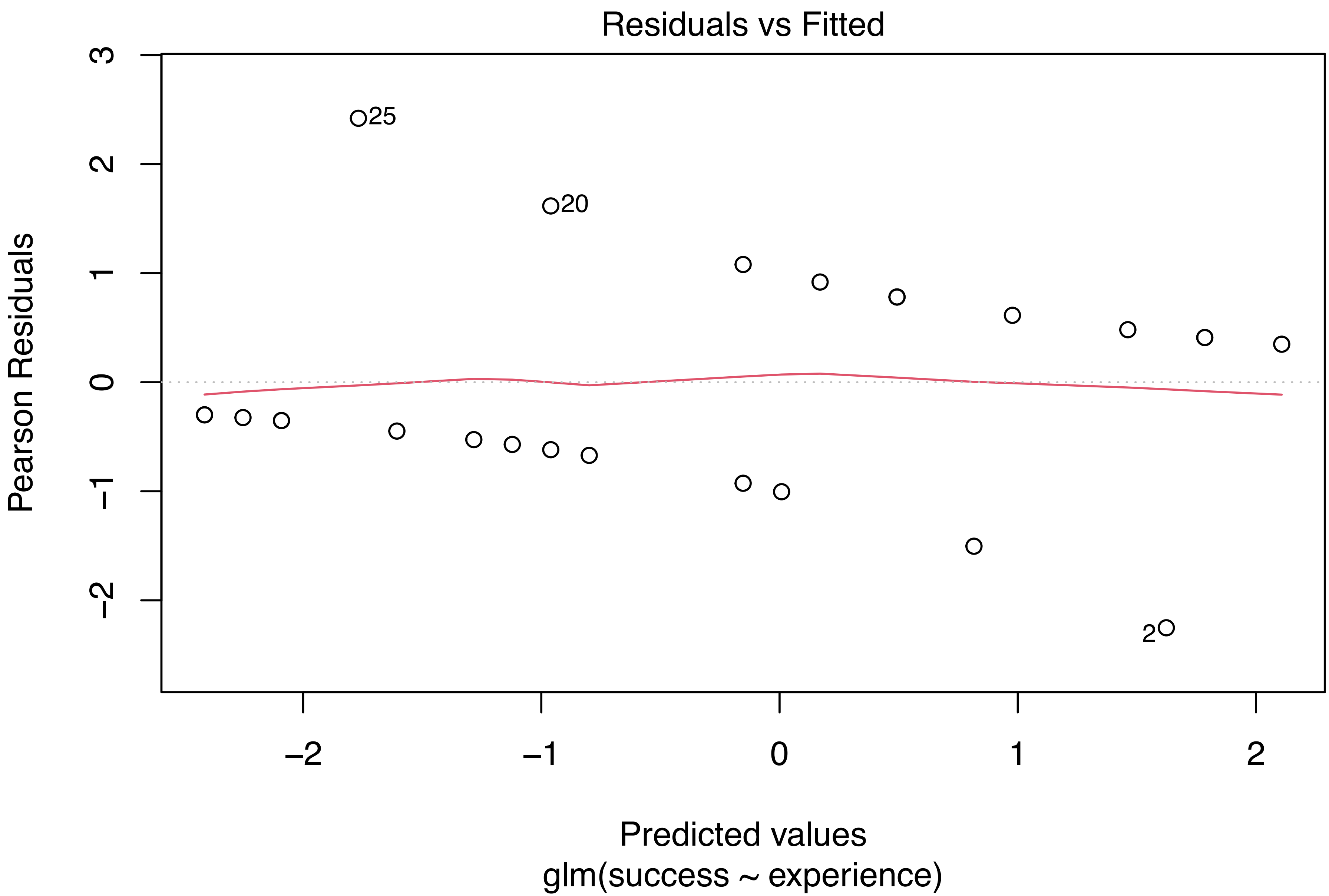
- Ordinary residuals $Y_i - \hat{\pi}_i$ cannot be Normally distributed.
- In GLMs, one looks at special residuals called deviance residuals.
- In logistic regression, the deviance residuals are defined as

$$\hat{d}_i = \operatorname{sign}(Y_i - \hat{\pi}_i)\sqrt{-2[Y_i \log \hat{\pi}_i + (1 - Y_i)\log(1 - \hat{\pi}_i)]}$$

  for $i = 1, \ldots, n$.
- These are not Normal either, but are useful for assessing model fit.

Programming task data (cont)

```
plot(glm_out,which = 1)
```



Checking model fit with a simulated envelope

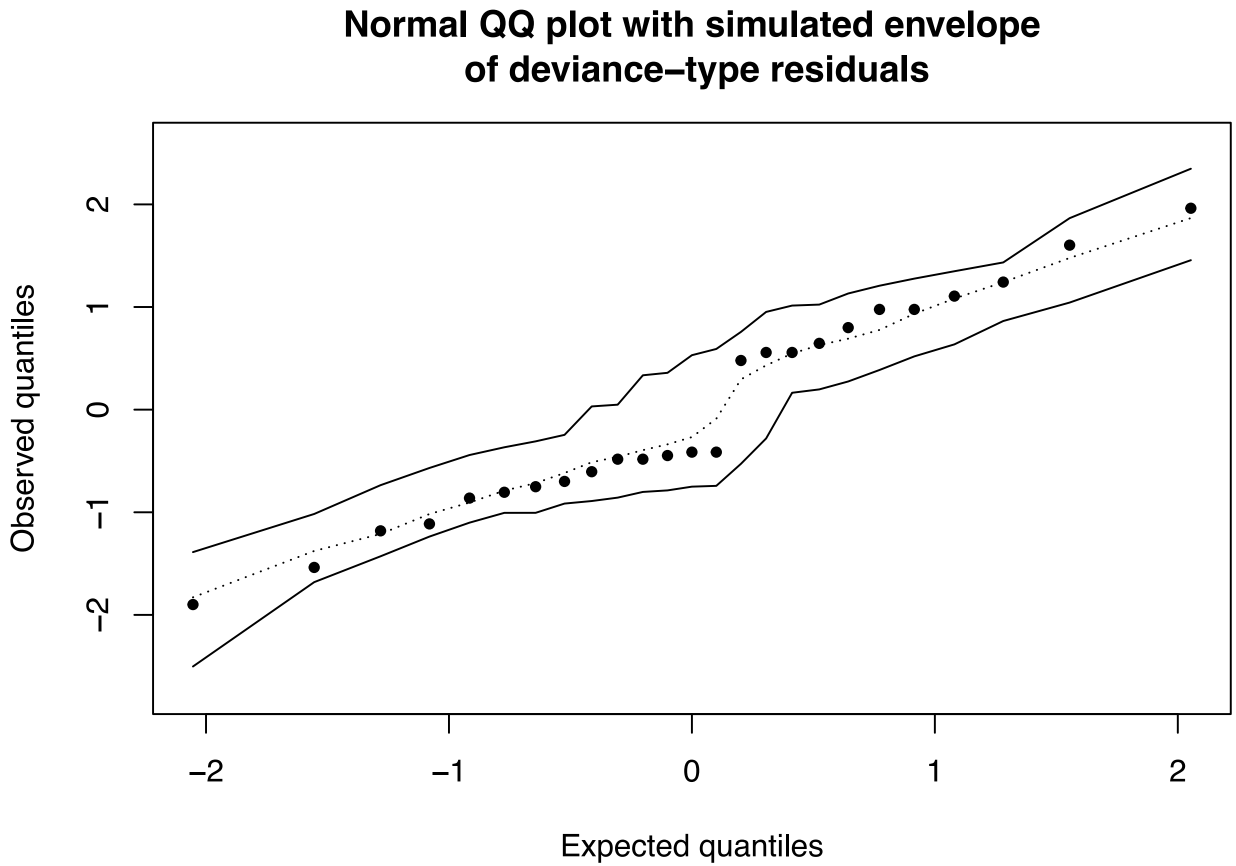
The simulated envelope method is described in Kutner et al. (2005):

- Fit the logistic regression model and obtain $\hat{\pi}_1, \dots, \hat{\pi}_n$.
- Obtain the deviance residuals; sort them as $\hat{d}_{(1)} < \hat{d}_{(2)} < \cdots < \hat{d}_{(n)}$.
- Generate many new data sets $Y_i^* \sim \text{Bernoulli}(\hat{\pi}_i)$, $i = 1, \dots, n$.
- For each new data set, obtain sorted $\hat{d}^*_{(1)} < \hat{d}^*_{(2)} < \cdots < \hat{d}^*_{(n)}$.
- Plot $\hat{d}_{(i)}$ as well as the 0.025 and 0.975 quantiles and the mean of the $\hat{d}^*_{(i)}$ $\forall\, i$ (it doesn't matter what is chosen as the x-axis).
- The quantiles of the $\hat{d}^*_{(i)}$ make a band. If the model fits, then the $\hat{d}_{(i)}$ should lie within the band and close to the mean.

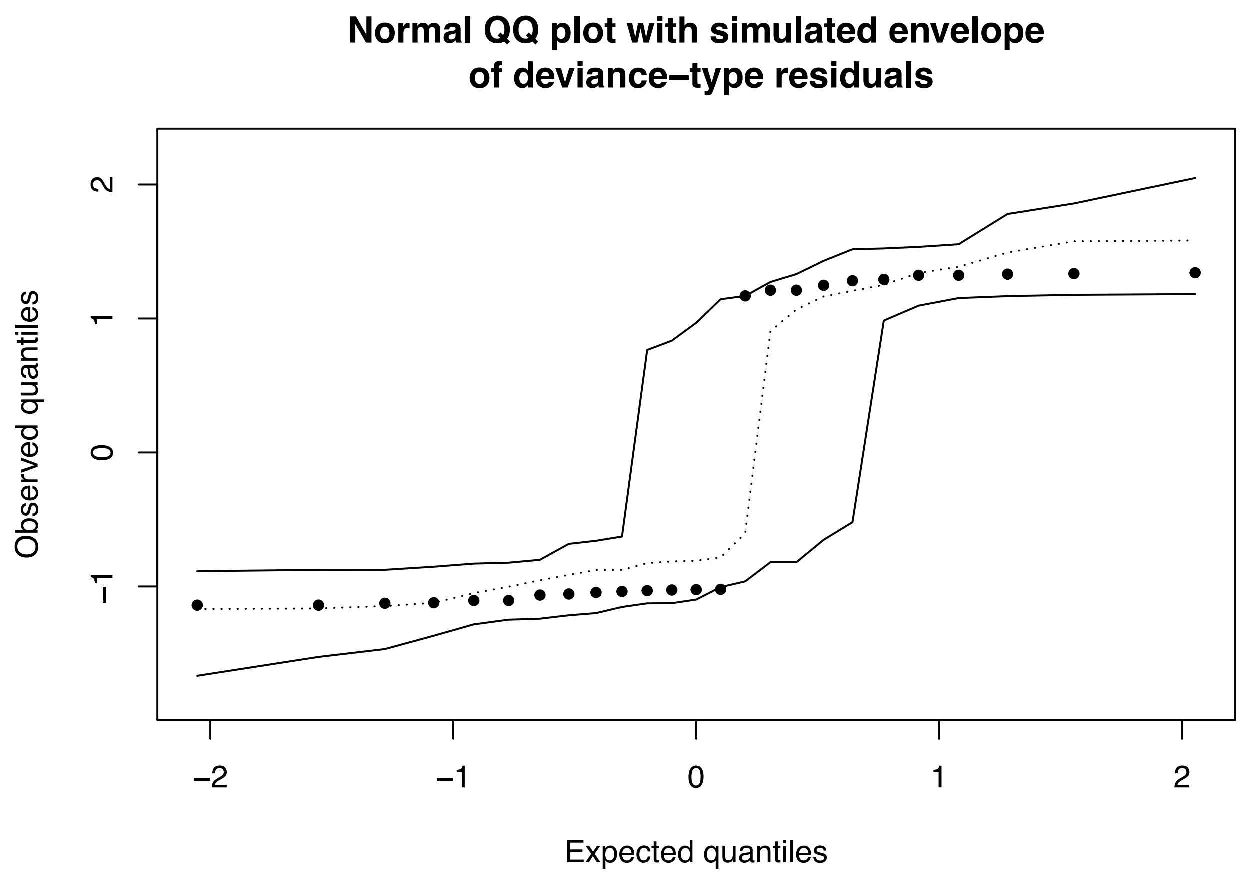
Asks: If the model is correct, how would the deviance residuals behave?

Programming task data (cont)

```
library(glmtoolbox) # first time run install.packages("glmtoolbox")
envelope(glm_out,type = "deviance")
```



```r
experience2 <- (experience - mean(experience))^2
envelope(glm(success ~ experience2, family = "binomial"),type = "deviance")
```



German credit score data from Hofmann (1994)

Response is credit rating (good/bad), various predictors.

```
library(foreign) # credit-g dataset from https://www.openml.org/
link <- url("https://people.stat.sc.edu/gregorkb/data/dataset_31_credit-g.arff")
credg <- read.arff(link)
colnames(credg)
```

```
 [1] "checking_status"        "duration"               "credit_history"
 [4] "purpose"                "credit_amount"          "savings_status"
 [7] "employment"             "installment_commitment" "personal_status"
[10] "other_parties"          "residence_since"        "property_magnitude"
[13] "age"                    "other_payment_plans"    "housing"
[16] "existing_credits"       "job"                    "num_dependents"
[19] "own_telephone"          "foreign_worker"         "class"
```

```r
summary(credg[,1:3])
```

```
   checking_status    duration                             credit_history
 <0         :274   Min.   : 4.0   all paid                        : 49
 >=200      : 63   1st Qu.:12.0   critical/other existing credit:293
 0<=X<200   :269   Median :18.0   delayed previously              : 88
 no checking:394   Mean   :20.9   existing paid                   :530
                   3rd Qu.:24.0   no credits/all paid             : 40
                   Max.   :72.0
```

Multiple logistic regression

Observe $(\underset{\sim}{x}_1, Y_1), \dots, (\underset{\sim}{x}_n, Y_n)$,

$$\underset{p\times 1}{\underset{\sim}{x}_i} = \begin{bmatrix} x_{i1} \\ \vdots \\ x_{ip} \end{bmatrix} \in \mathbb{R}^p \qquad Y_i \in \{0,1\}.$$

Assume

$Y_i \sim \text{Bernoulli}(\pi_i)$,

$$\log\left(\frac{\pi_i}{1-\pi_i}\right) = \beta_0 + \beta_1 x_{i1} + \cdots + \beta_p x_{ip}$$

How do we interpret β_j?

Suppose you have covariate vector $\underset{\sim}{x}_0 = \begin{bmatrix} x_{01} \\ \vdots \\ x_{0p} \end{bmatrix}$.

Increase covariate j by one unit to get

$$\underset{\sim}{x}_1 = \begin{bmatrix} x_{01} \\ \vdots \\ x_{0j}+1 \\ \vdots \\ x_{0p} \end{bmatrix}$$

$$\log\left(\frac{\pi_0}{1-\pi_0}\right) = \beta_0 + \beta_1 x_{01} + \cdots + \beta_j x_{0j} + \cdots + \beta_p x_{0p}$$

$$\log\left(\frac{\pi_1}{1-\pi_1}\right) = \beta_0 + \beta_1 x_{01} + \cdots + \beta_j (x_{0j}+1) + \cdots + \beta_p x_{0p}$$

$$\log\left(\frac{\pi_1}{1-\pi_1}\right) - \log\left(\frac{\pi_0}{1-\pi_0}\right) = \beta_j$$

$$\Leftrightarrow \quad \log\left(\frac{\frac{\pi_1}{1-\pi_1}}{\frac{\pi_0}{1-\pi_0}}\right) = \beta_j$$

$$\Leftrightarrow \quad \frac{\frac{\pi_1}{1-\pi_1}}{\frac{\pi_0}{1-\pi_0}} = e^{\beta_j}$$

Logistic multiple regression model

Observe $(\mathbf{x}_1, Y_1), \dots, (\mathbf{x}_n, Y_n)$, $\mathbf{x}_i = (x_{i1}, \dots, x_{ip})^T$ and assume

$$Y_i \sim \text{Bernoulli}(\pi_i), \quad \log\left(\frac{\pi_i}{1-\pi_i}\right) = \beta_0 + \beta_1 x_{i1} + \cdots + \beta_p x_{ip},$$

for $i = 1, \dots, n$, where

- Y_i is the response for observation i.
- $x_{i1}, \dots, x_{ip}$ are the values of the predictors for obs i.
- π_i is the probability of "success" for observation i.
- β_0 is an intercept and $\beta_1, \dots, \beta_p$ are slope parameters.
- $\pi_i/(1-\pi_i)$ is the odds of "success" for obs i.
- $\log(\pi_i/(1-\pi_i))$ is the log-odds for obs i.

So we assume the log-odds are a linear function of the predictors.

Interpreting multiple logistic regression parameters

- Let π_{0j} and π_{1j} be the "success" probabilities at x_{0j} and $x_{0j}+1$ but with x_{0k} fixed for all $k \neq j$.

- Then we have the two equations
  1. $\log\left(\frac{\pi_{0j}}{1-\pi_{0j}}\right) = \beta_0 + \sum_{k\neq j} \beta_k x_{0k} + \beta_j x_{0j}$
  2. $\log\left(\frac{\pi_{1j}}{1-\pi_{1j}}\right) = \beta_0 + \sum_{k\neq j} \beta_k x_{0k} + \beta_j (x_{0j}+1)$

- Subtracting the first equation from the second gives

$$\beta_j = \log\left(\frac{\pi_{1j}/(1-\pi_{1j})}{\pi_{0j}/(1-\pi_{0j})}\right) \quad \text{and} \quad e^{\beta_j} = \frac{\pi_{1j}/(1-\pi_{1j})}{\pi_{0j}/(1-\pi_{0j})}.$$

- So β_1 is log of the odds ratio associated with a unit increase in x_j with all other predictors held fixed.

German credit score data (cont)

```r
glm_out <- glm(class ~ ., family = "binomial", data = credg)
summary(glm_out)
```

```
Call:
glm(formula = class ~ ., family = "binomial", data = credg)

Coefficients:
                                       Estimate Std. Error z value Pr(>|z|)
(Intercept)                           1.505e+00  1.248e+00   1.206 0.227801
checking_status>=200                  9.657e-01  3.692e-01   2.616 0.008905 **
checking_status0<=X<200               3.749e-01  2.179e-01   1.720 0.085400 .
checking_statusno checking            1.712e+00  2.322e-01   7.373 1.66e-13 ***
duration                             -2.786e-02  9.296e-03  -2.997 0.002724 **
credit_historycritical/other existing credit  1.579e+00  4.381e-01   3.605 0.000312 ***
credit_historydelayed previously      9.965e-01  4.703e-01   2.119 0.034105 *
credit_historyexisting paid           7.295e-01  3.852e-01   1.894 0.058238 .
credit_historyno credits/all paid     1.434e-01  5.489e-01   0.261 0.793921
purposedomestic appliance            -2.173e-01  8.041e-01  -0.270 0.786976
purposeeducation                     -7.764e-01  4.660e-01  -1.666 0.095718 .
purposefurniture/equipment            5.152e-02  3.543e-01   0.145 0.884391
purposenew car                       -7.401e-01  3.339e-01  -2.216 0.026668 *
purposeother                          7.487e-01  7.998e-01   0.936 0.349202
purposeradio/tv                       1.515e-01  3.370e-01   0.450 0.653002
purposerepairs                       -5.237e-01  5.933e-01  -0.883 0.377428
purposeretraining                     1.319e+00  1.233e+00   1.070 0.284625
purposeused car                       9.264e-01  4.409e-01   2.101 0.035645 *
credit_amount                        -1.283e-04  4.444e-05  -2.887 0.003894 **
savings_status>=1000                  1.339e+00  5.249e-01   2.551 0.010729 *
savings_status100<=X<500              3.577e-01  2.861e-01   1.250 0.211130
savings_status500<=X<1000             3.761e-01  4.011e-01   0.938 0.348476
savings_statusno known savings        9.467e-01  2.625e-01   3.607 0.000310 ***
employment>=7                         2.097e-01  2.947e-01   0.712 0.476718
employment1<=X<4                      1.159e-01  2.423e-01   0.478 0.632415
employment4<=X<7                      7.641e-01  3.051e-01   2.504 0.012271 *
employmentunemployed                 -6.691e-02  4.270e-01  -0.157 0.875475
installment_commitment               -3.301e-01  8.828e-02  -3.739 0.000185 ***
```

Note that `glm()` estimates three coefficients for `checking_status`.

```
summary(credg$checking_status)
```

```
     <0   >=200 0<=X<200 no checking
    274      63      269         394
```

Numeric predictors to encode the levels of the categorical predictor:

$$x_{i1} = \begin{cases} 1 & 200 \le \texttt{checking} \\ 0 & \text{otherwise} \end{cases}$$

$$x_{i2} = \begin{cases} 1 & 0 \le \texttt{checking} < 200 \\ 0 & \text{otherwise} \end{cases}$$

$$x_{i3} = \begin{cases} 1 & \texttt{no checking} \\ 0 & \text{otherwise} \end{cases}$$

Likewise for other categorical predictors.

Deviances replace error sums of squares in GLMs

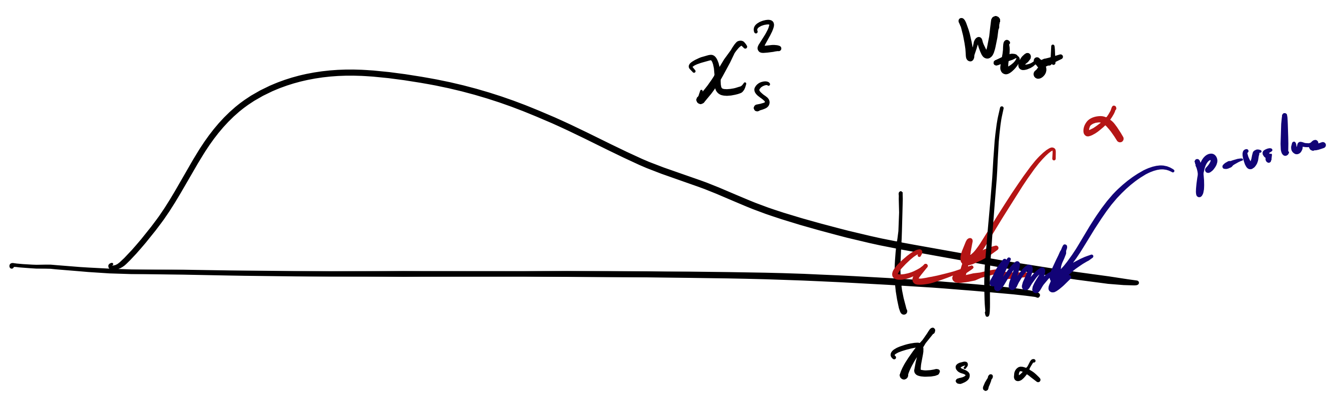
- The deviance is the sum of squared *deviance* residuals $\sum_{i=1}^{n} \hat{d}_i^2$.
- In logistic regression the deviance can be computed as

$$\text{Dev} = -2\sum_{i=1}^{n}[Y_i \log \hat{\pi}_i + (1 - Y_i)\log(1 - \hat{\pi}_i)].$$

- Full-reduced model test: Reject H_0: $\beta_j = 0$ for all $j \in D$ if

$$\text{Dev(Reduced)} - \text{Dev(Full)} > \chi^2_{s,\alpha},$$

where s is the number of predictors in D (need large n).



German credit score data (cont)

Test whether any level of checking status is important to the credit score.

```
credg_red <- credg[,-1] # remove checking_status column

glm_full <- glm(class ~ ., family = "binomial", data = credg)
glm_red <-  glm(class ~ ., family = "binomial", data = credg_red)

p <- length(coef(glm_full)) - 1
s <- nlevels(credg$checking_status) - 1

1 - pchisq(glm_red$deviance - glm_full$deviance,s)
```

```
[1] 2.731149e-14
```

Variable selection in logistic regression

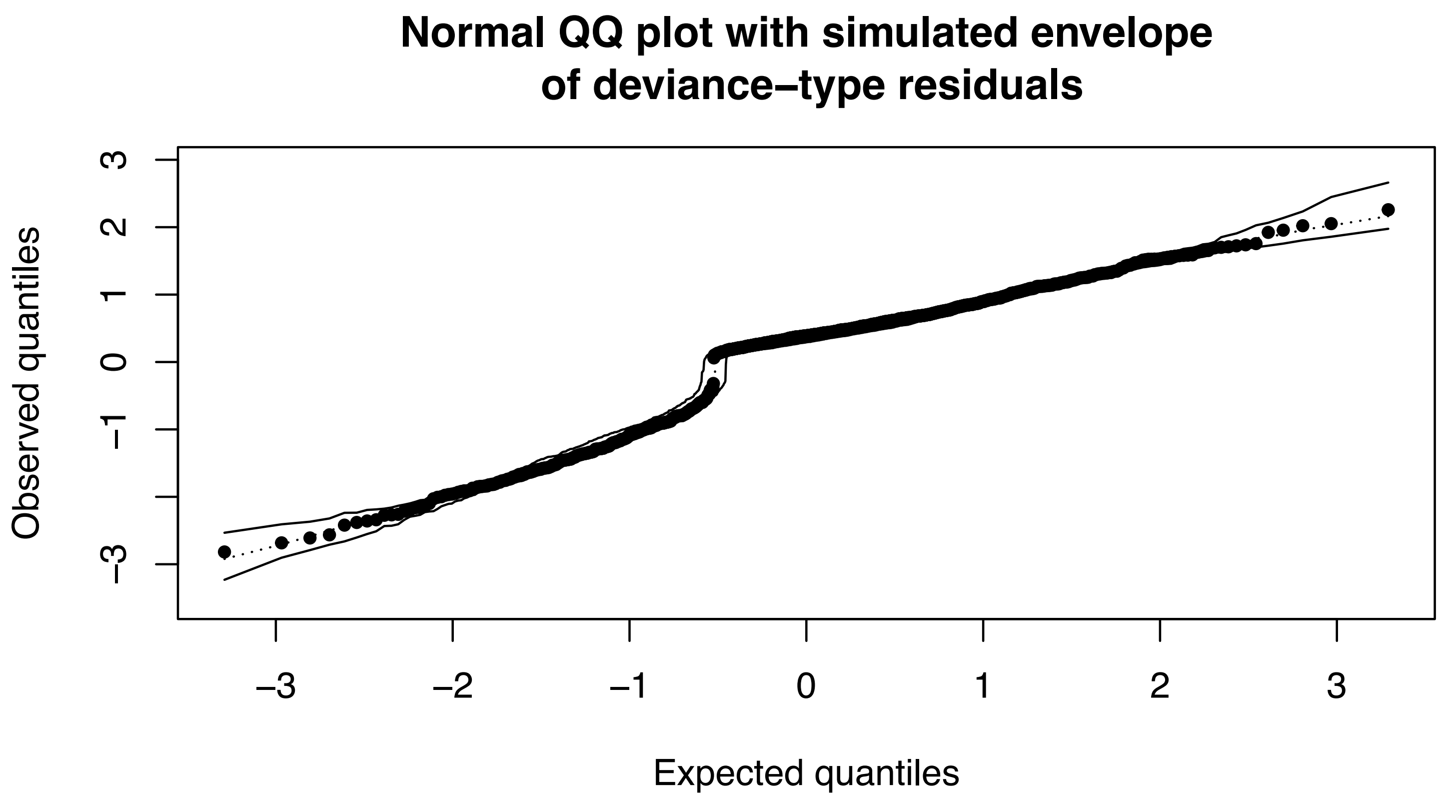
- We may want to discard some of our predictors.
- One way is to add/remove variables stepwise according to AIC.
- Can do this just as we did in multiple linear regression.
- Be cautious about making inferences after selecting a model.

German credit score data (cont)

```r
glm_all <- glm(class ~ ., family = "binomial", data = credg)
step_back <- step(glm_all,
                  direction = "backward",
                  scope = formula(glm_all),
                  criterion = "aic",
                  trace = 0) # suppress printed output
summary(step_back)
```

```
Call:
glm(formula = class ~ checking_status + duration + credit_history +
    purpose + credit_amount + savings_status + installment_commitment +
    personal_status + other_parties + age + other_payment_plans +
    housing + own_telephone + foreign_worker, family = "binomial",
    data = credg)

Coefficients:
                                              Estimate Std. Error z value Pr(>|z|)
(Intercept)                                  4.838e-01  1.017e+00   0.476 0.634362
checking_status>=200                         1.024e+00  3.626e-01   2.824 0.004739 **
checking_status0<=X<200                      3.900e-01  2.121e-01   1.839 0.065928 .
checking_statusno checking                   1.718e+00  2.281e-01   7.531 5.05e-14 ***
duration                                    -2.568e-02  8.940e-03  -2.872 0.004074 **
credit_historycritical/other existing credit 1.373e+00  4.041e-01   3.397 0.000680 ***
credit_historydelayed previously             7.910e-01  4.488e-01   1.762 0.077985 .
credit_historyexisting paid                  7.115e-01  3.788e-01   1.879 0.060305 .
credit_historyno credits/all paid           -1.188e-01  5.268e-01  -0.225 0.821612
purposedomestic appliance                   -2.576e-01  7.763e-01  -0.332 0.740041
purposeeducation                            -9.262e-01  4.569e-01  -2.027 0.042628 *
purposefurniture/equipment                  -4.216e-02  3.415e-01  -0.123 0.901748
purposenew car                              -7.827e-01  3.272e-01  -2.392 0.016752 *
purposeother                                 6.523e-01  7.832e-01   0.833 0.404946
purposeradio/tv                              1.368e-01  3.288e-01   0.416 0.677335
purposerepairs                              -6.402e-01  5.808e-01  -1.102 0.270365
purposeretraining                            1.382e+00  1.240e+00   1.114 0.265228
purposeused car                              8.246e-01  4.288e-01   1.923 0.054495 .
credit_amount                               -1.294e-04  4.221e-05  -3.066 0.002169 **
```

```r
envelope(step_back,type="deviance")
```



Classification with logistic regression

$$R^2 = \frac{SS_{Reg}}{SS_{Total}} = 1 - \frac{SS_{Error}}{SS_{Total}}$$

Fitted probabilities $\hat{\pi}_1, \ldots, \hat{\pi}_n$.

What about $\hat{Y}_1, \ldots, \hat{Y}_n$?

Define predicted value $\hat{Y}_i$ as

$$\hat{Y}_i = \begin{cases} 1 & \hat{\pi}_i \geq c \\ 0 & \hat{\pi}_i < c \end{cases}$$

for some $c \in [0, 1]$.

Misclassification rate: $\dfrac{\#\{\hat{Y}_i \neq Y_i\}}{n}$

True positive rate: $\dfrac{\#\{\hat{Y}_i = 1,\ Y_i = 1\}}{\#\{Y_i = 1\}}$

False positive rate: $\dfrac{\#\{\hat{Y}_i = 1,\ Y_i = 0\}}{\#\{Y_i = 0\}}$

Classification with the logistic regression model

Consider classifying the observations as 1 or 0 according to $\hat{\pi}_i$:

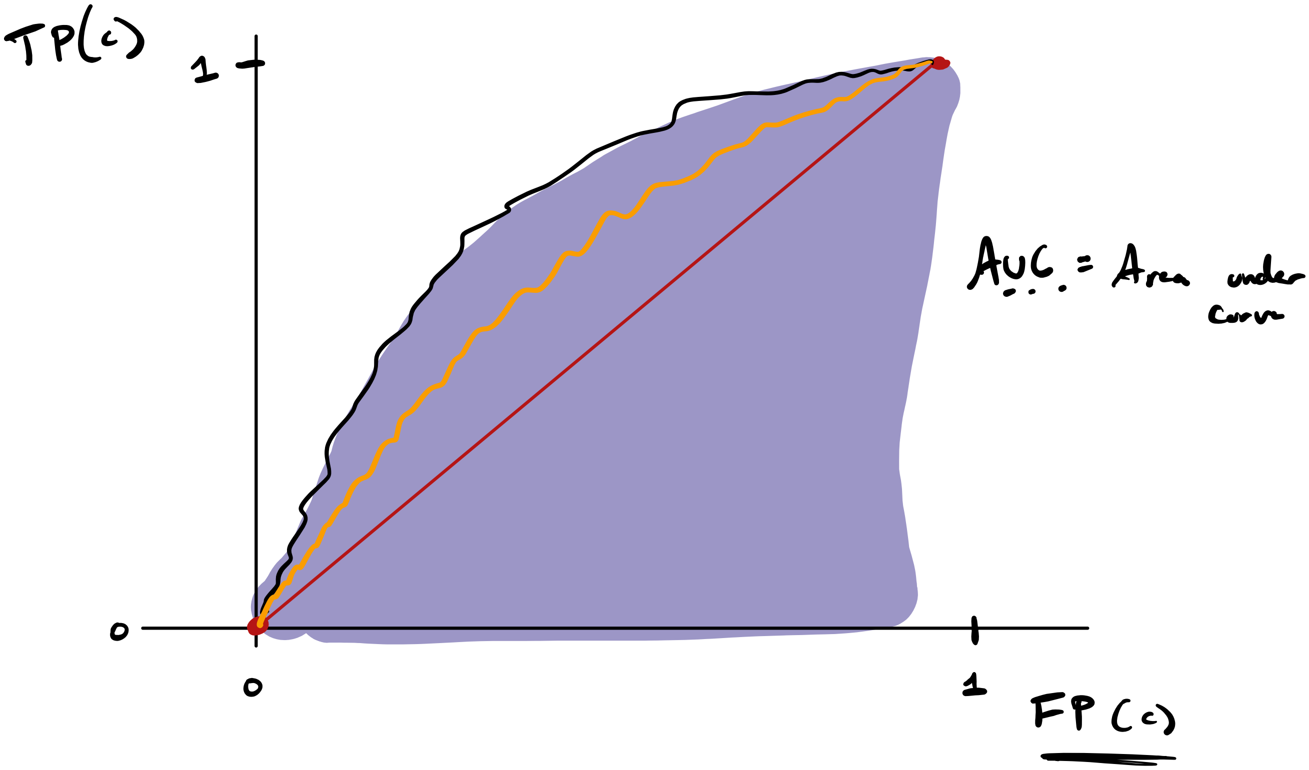
- Choose a threshold $c \in [0, 1]$ and make the classification

$$\hat{Y}_i = \begin{cases} 1, & \hat{\pi}_i \geq c \\ 0, & \hat{\pi}_i < c. \end{cases}$$

- Can compute misclassification rate: $\#\{\hat{Y}_i \neq Y_i\}/n$.
- Can compute observed true positive and false positive rates

$$\text{TP} = \frac{\#\{\hat{Y}_i = 1 \cap Y_i = 1\}}{\#\{Y_i = 1\}} \quad \text{and} \quad \text{FP} = \frac{\#\{\hat{Y}_i = 1 \cap Y_i = 0\}}{\#\{Y_i = 0\}}.$$

- Plotting TP against FP over all $c \in [0, 1]$ creates the receiver operating characteristic (ROC) curve.



$$\hat{Y}_i = \begin{cases} 1 & \text{if } \hat{\pi}_i \geq c \\ 0 & \text{if } \hat{\pi}_i < c \end{cases}$$

If $c = 1$. Then I never get $\hat{Y}_i = 1$

$TP(1) = 0$

$FP(1) = 0$

If $c = 0$. Then I always get $\hat{Y}_i = 1$

$TP(0) = 1$

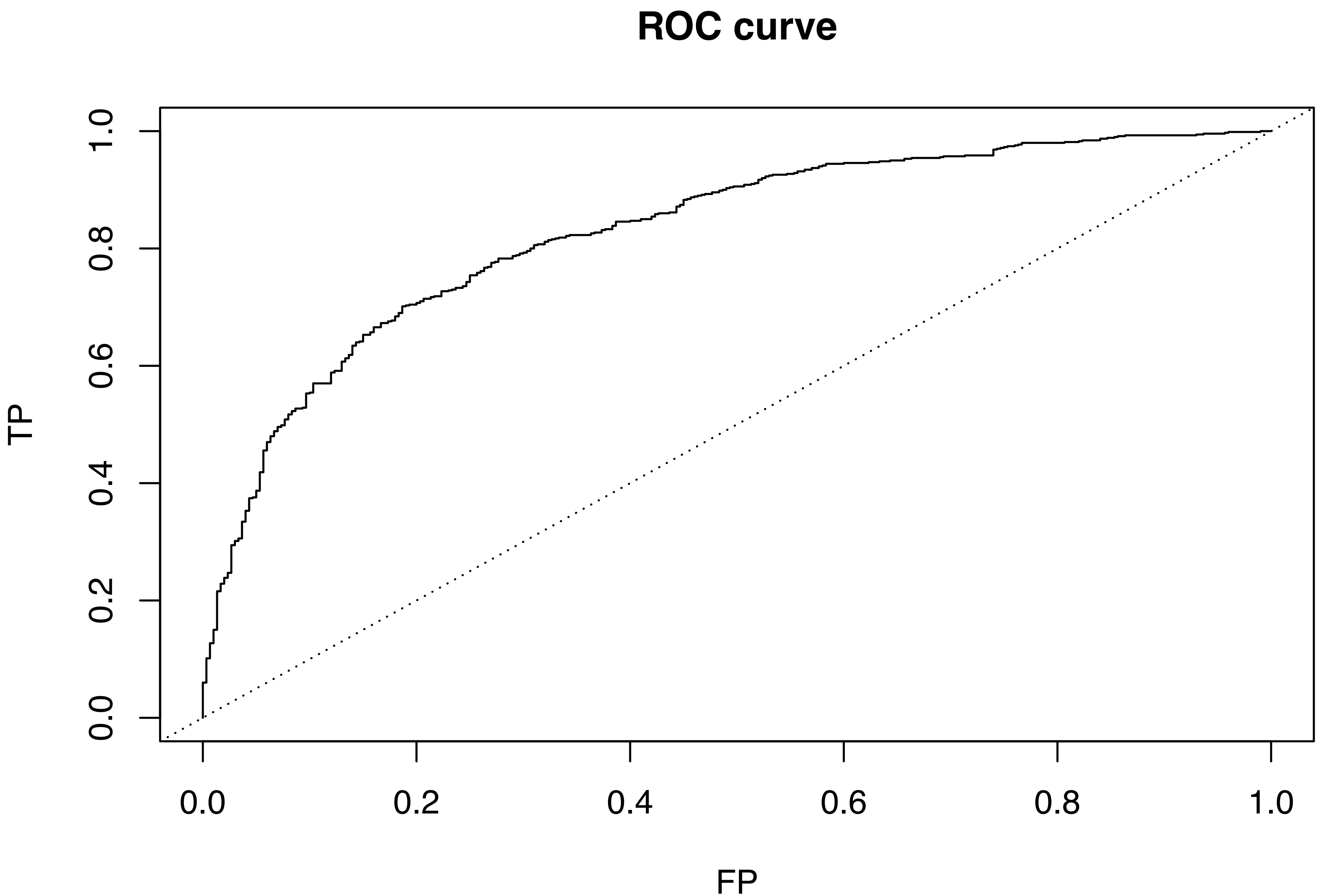
$FP(0) = 1$

German credit score data (cont)

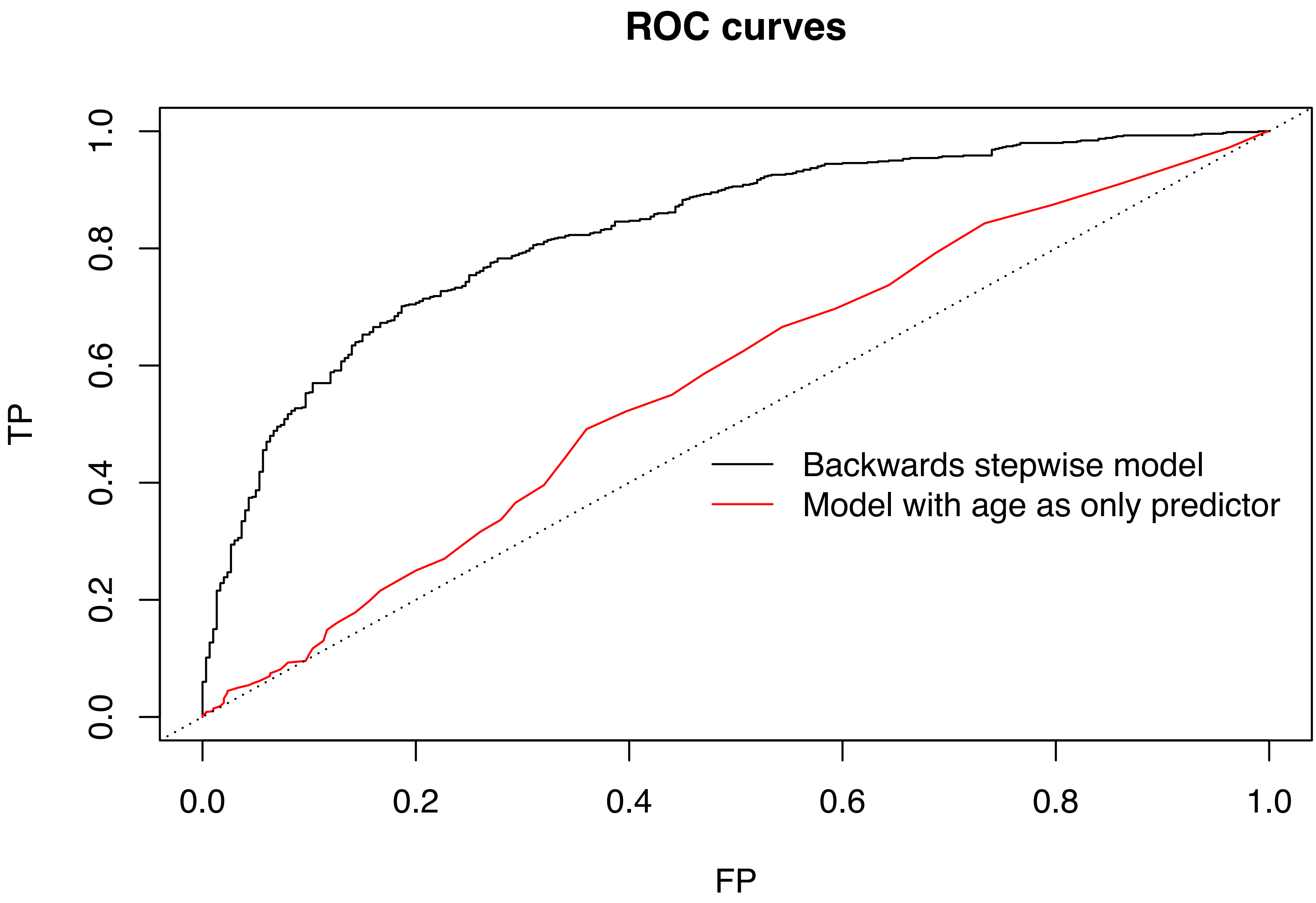
Compute TP and FP over a range of thresholds c. Plot ROC curve.

```
Y <- ifelse(credg$class == "good",1,0)
pi_hat <- step_back$fitted.values

n1 <- sum(Y == 1)
n0 <- sum(Y == 0)

cc <- sort(c(0,pi_hat,1))
TP <- FP <- numeric(length(cc))
for(j in 1:length(cc)){

  Yhat <- pi_hat >= cc[j]
  TP[j] <- sum(Yhat == 1 & Y == 1) / n1
  FP[j] <- sum(Yhat == 1 & Y == 0) / n0

}
```

```r
plot(TP ~ FP, type = "l", main = "ROC curve")
abline(0,1,lty = 3)
```

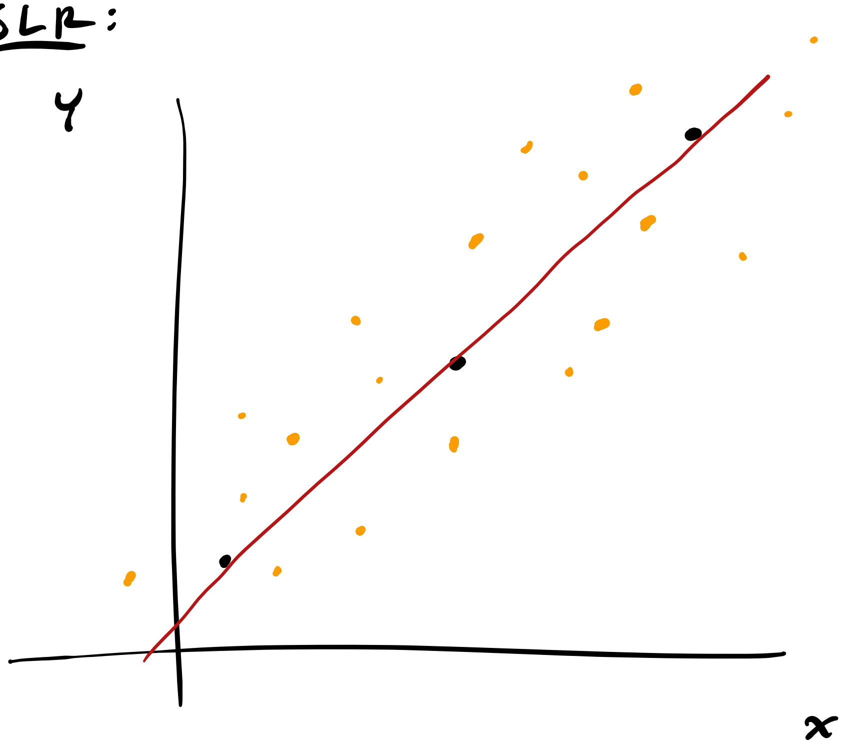


Can use ROC curves to compare models. [1]



[1] Best to evaluate model performance on a set of data not used in fitting the model.

SLR:



$R^2 \approx 1$

On new data, we might not perform as well.

Traning and testing data sets

- Overfitting is fitting a model such that it performs well on the data used to fit it, but will perform poorly when it is given new data.
- To avoid, split data into training and testing data sets

$$\{(\mathbf{x}_i^{\text{train}}, Y_i^{\text{train}})\}_{i \in \mathcal{I}_{\text{train}}} \quad \text{and} \quad \{(\mathbf{x}_i^{\text{test}}, Y_i^{\text{test}})\}_{i \in \mathcal{I}_{\text{test}}},$$

  where $\mathcal{I}_{\text{train}} \cap \mathcal{I}_{\text{test}} = \emptyset$ and $\mathcal{I}_{\text{train}} \cup \mathcal{I}_{\text{test}} = \{1, \dots, n\}$.
- Fit models on training data. Compare performance on testing data.
- For example, compute misclassification rate or draw ROC curve for testing data.

$\hat\beta_0, \hat\beta_1, \dots, \hat\beta_p$ from training data.

$$\hat\pi_{new} = \frac{e^{\hat\beta_0 + \hat\beta_1 x_{new,1} + \dots + \hat\beta_p x_{new,p}}}{1 + e^{\hat\beta_0 + \hat\beta_1 x_{new,1} + \dots + \hat\beta_p x_{new,p}}}$$

German credit score data (cont)

```
set.seed(1)
n <- nrow(credg)
n_train <- ceiling(.5*n)
n_test <- n - n_train
ind_train <- sample(n,n_train) # randomly select a training set

credg_train <- credg[ind_train,]
credg_test <- credg[-ind_train,]

glm_train <- glm(class ~ ., family = "binomial", data = credg_train)

ypred_train <- glm_train$fitted.values >= 0.5
y_train <- as.numeric(credg_train$class == "good")
misclass_train <- mean(ypred_train != y_train)
misclass_train
```

```
[1] 0.204
```

```
pihat_test <- predict(glm_train,newdata = credg_test,type="response")

ypred_test <- pihat_test >= 0.5
y_test <- as.numeric(credg_test$class == "good")

misclass_test <- mean(ypred_test != y_test)
misclass_test
```

```
[1] 0.258
```

References

Hofmann, Hans. 1994. "Statlog (German Credit Data)." UCI Machine Learning Repository.

Kutner, Michael H, Christopher J Nachtsheim, John Neter, and William Li. 2005. *Applied Linear Statistical Models*. McGraw-hill.