STAT 714 fa 2023 Exam 2

1. Let $Y_i = \mu + \varepsilon_i$, where $\varepsilon_i \overset{\text{ind}}{\sim} \text{Normal}(0, \sigma_i^2)$ for $i = 1, \dots, n$. Assume $\sigma_1^2, \dots, \sigma_n^2$ are known.
   (a) Give the generalized least squares estimator $\hat{\mu}_{\text{gls}}$ of μ.
   (b) Give the ordinary least squares estimator of $\hat{\mu}_{\text{ols}}$ of μ.
   (c) Give $\operatorname{Var} \hat{\mu}_{\text{gls}}$.
   (d) Give $\operatorname{Var} \hat{\mu}_{\text{ols}}$.
   (e) Compare $\operatorname{Var} \hat{\mu}_{\text{gls}}$ and $\operatorname{Var} \hat{\mu}_{\text{ols}}$ when $\sigma_i^2 = \sigma^2$ for all i.
   (f) Consider testing H_0: $\mu = 0$ versus H_1: $\mu \neq 0$ with the test statistic $\sum_{i=1}^n Y_i^2/\sigma_i^2$. Give the null distribution of the test statistic as well as its distribution under the alternate hypothesis.

2. Let $Y_{ij} = \mu_i + \beta_i x_{ij} + \varepsilon_{ij}$, $\varepsilon_{ij} \overset{\text{ind}}{\sim} \text{Normal}(0, \sigma^2)$ for $i = 1, 2, 3$ and $j = 1, \dots, n$. Assume for each i that $x_{i1}, \dots, x_{in}$ do not all take the same value.
   (a) Give the model in the matrix form $\mathbf{y} = \mathbf{X}\mathbf{b} + \mathbf{e}$. Give $\mathbf{X}$ and $\mathbf{b}$.
   (b) Give the smallest value of n such that the matrix $\mathbf{X}$ has full column rank.
   (c) Explain the purpose of the assumption that for each i, $x_{i1}, \dots, x_{in}$ do not all take the same value.
   (d) Give a matrix $\mathbf{K}$ and a vector $\mathbf{m}$ such that H_0: $\mathbf{K}^T\mathbf{b} = \mathbf{m}$ expresses H_0: $\beta_1 = \beta_2 = \beta_3 = 0$.
   (e) The rejection rule of the likelihood ratio test of H_0: $\beta_1 = \beta_2 = \beta_3 = 0$ can be calibrated using an F distribution. Give the numerator and denominator degrees of freedom of the relevant F distribution.
   (f) Suppose you wish to test whether the model $Y_{ij} = \mu + \beta x_{ij} + \varepsilon_{ij}$ is sufficient to describe the data (a single slope and intercept instead of distinct slopes and intercepts for $i = 1, 2, 3$). State the corresponding null hypothesis and give a matrix $\mathbf{K}$ and a vector $\mathbf{m}$ such that the hypothesis can be expressed as H_0: $\mathbf{K}^T\mathbf{b} = \mathbf{m}$.
   (g) Give the numerator and denominator degrees of freedom of the F distribution relevant to testing the hypothesis in (f).

3. Let $\mathbf{X}$ be an $n \times p$ matrix and $\mathbf{V}$ be a positive definite $p \times p$ matrix which is not the identity matrix. Recall that $(\mathbf{X}^T\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}^T\mathbf{V}^{-1}$ is a generalized inverse of $\mathbf{X}$.
   (a) Show that $\tilde{\mathbf{P}}_{\mathbf{X}} = \mathbf{X}(\mathbf{X}^T\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}^T\mathbf{V}^{-1}$ satisfies the three requirements of a projection onto $\operatorname{Col}\mathbf{X}$.
   (b) Argue carefully whether $\tilde{\mathbf{P}}_{\mathbf{X}}$ is an orthogonal projection onto $\operatorname{Col}\mathbf{X}$.