STAT 714 fa 2023 Final Exam

1. Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 3 & 1 & 2 \end{bmatrix}$.

   (a) Give a basis for $\operatorname{Col}\mathbf{A}$.

   (b) Give the rank of $\mathbf{A}^T\mathbf{A}$.

   (c) Give the minimum eigenvalue of $\mathbf{A}^T\mathbf{A}$.

   (d) Give the orthogonal projections of the vectors (i) $\mathbf{v} = (2, 3, 4)^T$ and (ii) $\mathbf{u} = (1, -2, 1)^T$ onto $\operatorname{Col}\mathbf{A}$.

2. Let $Y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$, $\varepsilon_i \overset{\text{ind}}{\sim} \text{Normal}(0, \sigma^2)$ for $i = 1, \dots, n$ and define the sums of squares

$$\text{SSE} = \sum_{i=1}^n (Y_i - \hat{Y}_i)^2, \quad \text{SSR} = \sum_{i=1}^n (\hat{Y}_i - \bar{Y}_n)^2, \quad \text{SST} = \sum_{i=1}^n (Y_i - \bar{Y}_n)^2,$$

   where $\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i$ for $i = 1, \dots, n$, where $\hat{\beta}_0$ and $\hat{\beta}_1$ are the least-squares estimators. Moreover, define $\hat{\varepsilon}_i = Y_i - \hat{Y}_i$ for $i = 1, \dots, n$.

   (a) Write the model in matrix form as $\mathbf{y} = \mathbf{X}\mathbf{b} + \mathbf{e}$.

   (b) Give the value of $\sum_{i=1}^n \hat{\varepsilon}_i x_i$. Show your work.

   (c) Give the value of $\sum_{i=1}^n \hat{\varepsilon}_i$. Show your work.

   (d) Write each of the sums of squares SSE, SSR, and SST as a quadratic form in $\mathbf{y}$.

   (e) Show that $\text{SST} = \text{SSR} + \text{SSE}$.

   (f) Give the distributions of the scaled sums of squares (i) SSE/σ^2 and (ii) SSR/σ^2.

   (g) Give a test of H_0: $\beta_1 = 0$ versus H_1: $\beta_1 \neq 0$ which has size α and which has power greater than α under the alternative. Make use of a test statistic which has an F distribution.

   (h) Describe the relationship between the quantities $\sum_{i=1}^n (x_i - \bar{x}_n)^2$, σ^2, and β_1^2 on the power of your test from part (g).

   (i) Find the REML estimator of σ^2 by maximizing the REML log-likelihood

$$\ell_R(\sigma^2; \mathbf{y}) = -\log|\mathbf{V}| - \log|\mathbf{X}^T\mathbf{V}^{-1}\mathbf{X}| - \mathbf{y}^T(\mathbf{V}^{-1} - \mathbf{V}^{-1}\mathbf{X}(\mathbf{X}^T\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}^T\mathbf{V}^{-1})\mathbf{y}.$$

3. Let $Y_{ij} = (\beta + B_i)x_{ij} + \varepsilon_{ij}$, $B_i \overset{\text{ind}}{\sim} \text{Normal}(0, \sigma_B^2)$, $\varepsilon_{ij} \overset{\text{ind}}{\sim} \text{Normal}(0, \sigma_\varepsilon^2)$, for $i = 1, \dots, a$ and $j = 1, \dots, n$, where β is a constant. Assume the B_i and the ε_{ij} are independent.

   (a) Write down the model in the form $\mathbf{y} = \mathbf{X}\mathbf{b} + \mathbf{Z}\mathbf{u} + \mathbf{e}$. It will be convenient to define the vectors $\mathbf{x}_i = (x_{i1}, \dots, x_{in})^T$ and $\mathbf{y}_i = (y_{i1}, \dots, y_{in})^T$ for $i = 1, \dots, a$.

   (b) Give the matrix $\mathbf{V} = \operatorname{Cov} \mathbf{y}$. Note that it should be a block-diagonal matrix.

   (c) Use the result $(a\mathbf{I}_n + b\mathbf{v}\mathbf{v}^T)^{-1} = \frac{1}{a}(\mathbf{I}_n - \frac{b}{a+b\|\mathbf{v}\|^2}\mathbf{v}\mathbf{v}^T)$ to find $\mathbf{V}^{-1}$.

   (d) Show that $\hat{\beta}_{\text{gls}} = \sum_{i=1}^a w_i \mathbf{y}_i^T \mathbf{x}_i / \sum_{i=1}^a w_i \|\mathbf{x}_i\|^2$, where $w_i = (\sigma_\varepsilon^2 + \sigma_B^2 \|\mathbf{x}_i\|^2)^{-1}$.

   (e) Find τ_i such that the BLUP for $v_i = \beta + B_i$ is given by

$$\tilde{v}_i = \tau_i(\mathbf{x}_i^T \mathbf{y}_i / \|\mathbf{x}_i\|^2) + (1 - \tau_i)\hat{\beta}_{\text{gls}}.$$

   Begin with the formula $\tilde{v} = \mathbf{c}^T \hat{\mathbf{b}}_{\text{gls}} + \mathbf{d}^T \mathbf{G}\mathbf{Z}^T \mathbf{V}^{-1}(\mathbf{y} - \mathbf{X}\hat{\mathbf{b}}_{\text{gls}})$ for the BLUP of $v = \mathbf{c}^T \mathbf{b} + \mathbf{d}^T \mathbf{u}$.