STAT 714 fa 2025 Exam 1

1. Consider the matrix
$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 3 & 2 \\ 0 & 1 & 2 & 0 \\ 1 & 2 & 5 & 2 \end{bmatrix}.$$
   (a) Give a basis for $\operatorname{Col}\mathbf{A}$.
   (b) Give a basis for $\operatorname{Nul}\mathbf{A}$.
   (c) Give a basis for the orthogonal complement of $\operatorname{Col}\mathbf{A}$.
   (d) For the vector
$$\mathbf{y} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$
   give $\hat{\mathbf{y}} \in \operatorname{Col}\mathbf{A}$ and $\hat{\mathbf{e}} \in (\operatorname{Col}\mathbf{A})^\perp$ such that $\mathbf{y} = \hat{\mathbf{y}} + \hat{\mathbf{e}}$.
   (e) Give the value of $\inf\limits_{\|\mathbf{x}\|=1} \mathbf{x}^T\mathbf{A}^T\mathbf{A}\mathbf{x}$.

2. Show that if $\mathbf{A}$ has full column rank we have $\mathbf{AB} = \mathbf{AC} \implies \mathbf{B} = \mathbf{C}$.

3. Let $\mathbf{A}$ be a symmetric matrix and let $\mathbf{G}$ be a generalized inverse of $\mathbf{A}$. Show that $(1/2)(\mathbf{G} + \mathbf{G}^T)$ is a generalized inverse of $\mathbf{A}$.

4. Show that the matrix $\mathbf{I}_n - \frac{1}{n}\mathbf{1}_n\mathbf{1}_n^T$ is a projection matrix, where $\mathbf{I}_n$ is the $n \times n$ identity matrix and $\mathbf{1}_n$ is the $n \times 1$ vector with every entry equal to 1.

5. Consider the analysis of covariance model given by
$$Y_{ij} = \mu + \alpha_i + \beta(x_{ij} - \bar{x}_{i.}) + \varepsilon_{ij}$$
for $i = 1, 2$, $j = 1, \dots, n$, where the ε_{ij} are independent random variables with mean zero and the x_{ij} are fixed covariate values with $\bar{x}_{i.} = n^{-1}\sum_{j=1}^n x_{ij}$ for $i = 1, 2$. Define
$$\mathbf{y} = \begin{bmatrix} Y_{11} \\ \vdots \\ Y_{1n} \\ Y_{21} \\ \vdots \\ Y_{2n} \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} \mathbf{1}_n & \mathbf{1}_n & \mathbf{0} & \mathbf{x}_1 \\ \mathbf{1}_n & \mathbf{0} & \mathbf{1}_n & \mathbf{x}_2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \beta \end{bmatrix}, \quad \text{and} \quad \mathbf{e} = \begin{bmatrix} \varepsilon_{11} \\ \vdots \\ \varepsilon_{1n} \\ \varepsilon_{21} \\ \vdots \\ \varepsilon_{2n} \end{bmatrix},$$

where

$$\mathbf{x}_i = \begin{bmatrix} x_{i1} - \bar{x}_{i.} \\ \vdots \\ x_{in} - \bar{x}_{i.} \end{bmatrix}$$

for $i = 1, 2$, so that the model can be written $\mathbf{y} = \mathbf{X}\mathbf{b} + \mathbf{e}$. Moreover, define the quantities

$$\bar{Y}_{..} = \frac{1}{2n} \sum_{i=1}^{2} \sum_{j=1}^{n} Y_{ij} \quad \text{and} \quad \bar{Y}_{i.} = \frac{1}{n} \sum_{j=1}^{n} Y_{ij}, \quad \text{for } i = 1, 2,$$

as well as

$$S_{xx} = \sum_{i=1}^{2} \sum_{j=1}^{n} (x_{ij} - \bar{x}_{i.})^2 \quad \text{and} \quad S_{xy} = \sum_{i=1}^{2} \sum_{j=1}^{n} Y_{ij}(x_{ij} - \bar{x}_{i.}).$$

(a) Check whether the following are estimable contrasts in $\mathbf{b}$:

  i. $\mu + \alpha_1$

  ii. μ

(b) Give $\mathbf{X}^T\mathbf{X}$.

(c) Give $\mathbf{X}^T\mathbf{y}$.

(d) Give a matrix $\mathbf{C}$ such that $\mathbf{C}\mathbf{b} = \mathbf{0}$ imposes the constraint $\alpha_1 + \alpha_2 = 0$.

(e) Write the solution to the equations

$$\begin{bmatrix} \mathbf{X}^T\mathbf{X} \\ \mathbf{C} \end{bmatrix} \mathbf{b} = \begin{bmatrix} \mathbf{X}^T\mathbf{y} \\ \mathbf{0} \end{bmatrix}$$

in terms of the quantities $\bar{Y}_{..}$, $\bar{Y}_{i.}$, $i = 1, 2$, S_{xx}, and S_{xy}.