STAT 714 fa 2025 Exam 1 solutions

$\boxed{1}$ (a) We have

$$A = \begin{bmatrix} 1 & 1 & 3 & 2 \\ 0 & 1 & 2 & 0 \\ 1 & 2 & 5 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

so the first two columns of A are pivot columns.

Therefore

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\}$$

is a basis for Col A.

(b) From the RREF of A we see that $A\underset{\sim}{x} = \underset{\sim}{0}$ has solution set

$$\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad x_3, x_4 \in \mathbb{R} \right\},$$

so

$$\left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is a basis for Nul A.

(c) Write

$$A^T = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 3 & 2 & 5 \\ 2 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

from which we see $A^T \underset{\sim}{x} = \underset{\sim}{0}$ has solution set

$$\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \quad x_3 \in \mathbb{R} \right\}$$

so that

$$\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

is a basis for $\text{Nul}\, A^T = (\text{Col}\, A)^\perp$.

(d) Write

$$\hat{\underset{\sim}{e}} = \text{Proj}_{(\text{Col}\, A)^\perp} \underset{\sim}{y} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \frac{\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}^T \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}}{\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}^T \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \frac{1}{3} = \begin{bmatrix} -1/3 \\ -1/3 \\ 1/3 \end{bmatrix}$$

Then

$$\hat{\underset{\sim}{y}} = \underset{\sim}{y} - \hat{\underset{\sim}{e}} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} - \begin{bmatrix} -1/3 \\ -1/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 4/3 \\ -5/3 \\ -1/3 \end{bmatrix}.$$

(e) This is the minimum eigenvalue of $A^T A$.

Since rank $A = 2$ we have rank $A^T A = 2$.

Since $A^T A$ is symm, # nonzero eigenvalues = rank.

So one eigenvalue is equal to zero.

So $\inf_{\|\underset{\sim}{x}\| = 1} \underset{\sim}{x}^T A^T A \underset{\sim}{x} = 0.$

$\boxed{2}$ First write

$$AB = AC \Rightarrow A(B - C) = 0.$$

Now if A has full column rank, Null $A = \{\underset{\sim}{0}\}$.

So the above implies $B - C = 0 \Rightarrow B = C$.

$\boxed{3}$ First, if A is symmetric, G^T is a g-inv of A^T, since

$$A G^T A = (A^T G A^T)^T = (AGA)^T = A^T = A$$

So we have

$$A \tfrac{1}{2}(G + G^T) A = \tfrac{1}{2} AGA + \tfrac{1}{2} AG^T A$$

$$= \tfrac{1}{2} A + \tfrac{1}{2} A$$

$$= A.$$

$\boxed{4}$ Since every idempotent matrix is a projection onto its own column space, we have only to show that the matrix is idempotent.

$$\left(I_n - \frac{1}{n}\mathbf{1}_n\mathbf{1}_n^T\right)\left(I_n - \frac{1}{n}\mathbf{1}_n\mathbf{1}_n^T\right) = I - \frac{1}{n}\mathbf{1}_n\mathbf{1}_n^T - \frac{1}{n}\mathbf{1}_n\mathbf{1}_n^T + \frac{1}{n}\mathbf{1}_n\underbrace{\mathbf{1}_n^T\mathbf{1}_n}_{n}\mathbf{1}_n^T\frac{1}{n}$$

$$= I - \frac{1}{n}\mathbf{1}_n\mathbf{1}_n^T$$

$\boxed{5}$ (a)

$$X^T = \begin{bmatrix} \mathbf{1}_n^T & \mathbf{1}_n^T \\ \mathbf{1}_n^T & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_n^T \\ \mathbf{x}_1^T & \mathbf{x}_2^T \end{bmatrix}$$

(i) $\mu + \alpha_1 = \mathbf{c}^T\mathbf{b}$ with $\mathbf{c} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ is estimable.

We may obtain $\mathbf{c}$ by adding the first n columns of X^T and then dividing by n, since $\mathbf{1}^T\mathbf{x}_1 = 0$. I.e.:

$$\mathbf{c} = X^T\begin{bmatrix} \frac{1}{n}\mathbf{1}_n \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \frac{1}{n}\mathbf{1}_n^T\mathbf{1}_n \\ \frac{1}{n}\mathbf{1}_n^T\mathbf{1}_n \\ \frac{1}{n}\mathbf{0}^T\mathbf{1}_n \\ \frac{1}{n}\mathbf{x}_1^T\mathbf{1}_n \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

(ii) $\mu = \mathbf{c}^T\mathbf{b}$ with $\mathbf{c} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ is not estimable.

No linear combination of the columns of X^T will yield $\mathbf{c}$.

(b)

$$X^T X = \begin{bmatrix} 2n & n & n & 0 \\ n & n & 0 & 0 \\ n & 0 & n & 0 \\ 0 & 0 & 0 & S_{xx} \end{bmatrix}$$

(c)

$$X^T \underset{\sim}{y} = \begin{bmatrix} 2n\,\bar{y}_{..} \\ n\,\bar{y}_{1.} \\ n\,\bar{y}_{2.} \\ S_{xy} \end{bmatrix}$$

(d) $C = \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}$

(e)

$$\left[\begin{array}{c|c} X^T X & X^T \underset{\sim}{y} \\ C & 0 \end{array}\right] = \left[\begin{array}{cccc|c} 2n & n & n & 0 & 2n\,\bar{y}_{..} \\ n & n & 0 & 0 & n\,\bar{y}_{1.} \\ n & 0 & n & 0 & n\,\bar{y}_{2.} \\ 0 & 0 & 0 & S_{xx} & S_{xy} \\ 0 & 1 & 1 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \bar{y}_{..} \\ 1 & 1 & 0 & 0 & \bar{y}_{1.} \\ 1 & 0 & 1 & 0 & \bar{y}_{2.} \\ 0 & 0 & 0 & 1 & S_{xy}/S_{xx} \\ 0 & 1 & 1 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \bar{Y}_{..} \\ 0 & 1 & 0 & 0 & \bar{Y}_{1.} - \bar{Y}_{..} \\ 0 & 0 & 1 & 0 & \bar{Y}_{2.} - \bar{Y}_{..} \\ 0 & 0 & 0 & 1 & S_{xy}/S_{xx} \\ 0 & 1 & 1 & 0 & 0 \end{array}\right]$$

The unique solution is given by

$$\hat{\underset{\sim}{b}} = \begin{bmatrix} \hat{\mu} \\ \hat{\alpha}_1 \\ \hat{\alpha}_2 \\ \hat{\beta} \end{bmatrix}, \quad \text{with} \quad \begin{aligned} \mu &= \bar{Y}_{..} \\ \hat{\alpha}_1 &= \bar{Y}_{1.} - \bar{Y}_{..} \\ \hat{\alpha}_2 &= \bar{Y}_{2.} - \bar{Y}_{..} \\ \hat{\beta} &= S_{xy}/S_{xx} \end{aligned}$$