STAT 714 fa 2025 Final Exam

1. Let $Y_t = \beta_0 + \beta_1 t + \varepsilon_t$ for $t = -T, \dots, T$, where $\varepsilon_{-T}, \dots, \varepsilon_T$ are independent $\mathcal{N}(0, 1)$ random variables.
   (a) Give $\mathbf{y}$, $\mathbf{X}$, $\mathbf{b}$, and $\mathbf{e}$ such that the model can be expressed as $\mathbf{y} = \mathbf{X}\mathbf{b} + \mathbf{e}$.
   (b) Give the matrix $\mathbf{X}^T\mathbf{X}$. *Hint:* $\sum_{i=1}^n i^2 = n(2n+1)(n+1)/6$.
   (c) Give the least squares estimator $\hat{\mathbf{b}}$ of $\mathbf{b}$.
   (d) Give the joint distribution of the entries $\hat{\beta}_0$ and $\hat{\beta}_1$ of $\hat{\mathbf{b}}$.
   (e) Give $\mathbf{K}$ and $\mathbf{m}$ such that $\mathbf{K}^T\mathbf{b} = \mathbf{m}$ if and only if $\beta_1 = 0$.
   (f) Under your choice of $\mathbf{K}$ and $\mathbf{m}$, give an expression for the quantity

   $$W = (\mathbf{K}^T\hat{\mathbf{b}} - \mathbf{m})^T[\mathbf{K}^T(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{K}]^{-1}(\mathbf{K}^T\hat{\mathbf{b}} - \mathbf{m})$$

   in terms of $Y_{-T}, \dots, Y_T$.
   (g) State exactly the distribution of W.
   (h) Propose a size-α test of H_0: $\beta_1 = 0$.

2. Let V and W be vector spaces. Show the following:
   (a) $V \subset W \implies W^\perp \subset V^\perp$.
   (b) $W \cap W^\perp = \{\mathbf{0}\}$.

3. (a) Let λ be an eigenvalue of $\mathbf{A}$. Show that if $\mathbf{A}$ is idempotent then λ is equal to 0 or 1.
   (b) Let $\mathbf{A}$ be an $n \times n$ symmetric, idempotent matrix with rank s. Argue that there exists an $n \times s$ matrix $\mathbf{G}$ such that $\mathbf{A} = \mathbf{G}\mathbf{G}^T$ and $\mathbf{G}^T\mathbf{G} = \mathbf{I}_s$.

4. Let $Y_{ij} = \mu + A_i + \sigma_i \varepsilon_{ij}$ for $i = 1, \dots, a$ and $j = 1, \dots, n$, where $A_1, \dots, A_a \overset{\text{ind}}{\sim} \mathcal{N}(0, \sigma_A^2)$ and $\varepsilon_{ij} \overset{\text{ind}}{\sim} \mathcal{N}(0, 1)$.
   (a) Give $\mathbf{y}$, $\mathbf{X}$, $\mathbf{b}$, $\mathbf{Z}$, $\mathbf{u}$, and $\mathbf{e}$ such that the model can be expressed as $\mathbf{y} = \mathbf{X}\mathbf{b} + \mathbf{Z}\mathbf{u} + \mathbf{e}$.
   (b) Give the matrix $\mathbf{V} = \operatorname{Cov} \mathbf{y}$.
   (c) Give the matrix $\mathbf{V}^{-1}$, using $(a\mathbf{J}_n + b\mathbf{I}_n)^{-1} = \frac{1}{b}(\mathbf{I}_n - \frac{a}{b+na}\mathbf{J}_n)$.
   (d) Give the generalized least-squares estimator $\hat{\mu}_{\text{gls}}$ for μ.
   (e) Give the BLUP for $v_i = \mu + A_i$, using the fact that the BLUP for $v = \mathbf{c}^T\mathbf{b} + \mathbf{d}^T\mathbf{u}$ is given by $\tilde{v} = \mathbf{c}^T\hat{\mathbf{b}}_{\text{gls}} + \mathbf{d}^T\mathbf{G}\mathbf{Z}^T\mathbf{V}^{-1}(\mathbf{y} - \mathbf{X}\hat{\mathbf{b}}_{\text{gls}})$.