STAT 714 fa 2025 final exam sol

1 $Y_t = \beta_0 + \beta_1 t + \varepsilon_t$, $t = -T, \ldots, T$, $\varepsilon_t \overset{ind}{\sim} N(0,1)$.

(a)
$$\underset{\sim}{Y} = \begin{bmatrix} Y_{-T} \\ \vdots \\ Y_T \end{bmatrix} \quad X = \begin{bmatrix} 1 & -T \\ \vdots & \vdots \\ 1 & T \end{bmatrix} \quad \underset{\sim}{b} = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} \quad \underset{\sim}{e} = \begin{bmatrix} \varepsilon_{-T} \\ \vdots \\ \varepsilon_T \end{bmatrix}$$

(b)
$$X^T X = \begin{bmatrix} 1 & -T \\ \vdots & \vdots \\ 1 & T \end{bmatrix}^T \begin{bmatrix} 1 & -T \\ \vdots & \vdots \\ 1 & T \end{bmatrix} = \begin{bmatrix} 2T+1 & 0 \\ 0 & \frac{T(2T+1)(T+1)}{3} \end{bmatrix}$$

(c)
$$\hat{\underset{\sim}{b}} = (X^T X)^{-1} X^T \underset{\sim}{Y}$$

$$= \begin{bmatrix} \frac{1}{2T+1} & 0 \\ 0 & \frac{3}{T(2T+1)(T+1)} \end{bmatrix} \begin{bmatrix} \sum_{t=-T}^{T} Y_t \\ \sum_{t=-T}^{T} t Y_t \end{bmatrix}$$

$$= \begin{bmatrix} \bar{Y}_\cdot \\ \frac{3}{T(2T+1)(T+1)} \sum_{t=-T}^{T} t Y_t \end{bmatrix} , \quad \bar{Y}_\cdot = \frac{1}{2T+1} \sum_{t=-T}^{T} t Y_t$$

(d)
$$\hat{\underset{\sim}{b}} \sim N\left(\underset{\sim}{b}, (X^T X)^{-1} \right)$$

$$= N\left(\begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix}, \begin{pmatrix} \frac{1}{2T+1} & 0 \\ 0 & \frac{3}{T(2T+1)(T+1)} \end{pmatrix} \right)$$

(e) With $K = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $m = [0]$ we have

$$K^T \underset{\sim}{b} = m \iff \beta_1 = 0.$$

(f)

$$W = \left(K^T \hat{\underset{\sim}{b}} - m\right)^T \left[K^T (X^T X)^{-1} K\right]^{-1} \left(K \hat{\underset{\sim}{b}} - m\right)$$

$$= \frac{T(2T+1)(T+1)}{3} \hat{\beta}_1^2$$

$$= \frac{3}{T(2T+1)(T+1)} \left(\sum_{t=-T}^{T} t \, Y_t \right)^2$$

(g) We have

$$W \sim \chi_1^2 \left(\phi = \frac{T(2T+1)(T+1)}{3} \beta_1^2 \right).$$

We obtain this by observing

$$K^T \hat{\underset{\sim}{b}} \sim N\left(K^T \underset{\sim}{b}, \; K^T (X^T X)^{-1} K\right),$$

from which we have

$$\left(K^T \hat{\underset{\sim}{b}}\right)^T \left[K (X^T X)^{-1} K\right]^{-1} K^T \hat{\underset{\sim}{b}} \sim \chi_1^2 \left(\phi = \left(K^T \underset{\sim}{b}\right)^T \left[K^T (X^T X)^{-1} K\right]^{-1} K^T \underset{\sim}{b} \right),$$

where $\left(K^T \underset{\sim}{b}\right)^T \left[K^T (X^T X)^{-1} K\right]^{-1} K^T \underset{\sim}{b} = \frac{T(2T+1)(T+1)}{3} \beta_1^2.$

(h) Reject $H_0: \beta_1 = 0$ if

$$\frac{3}{T(2T+1)(T+1)} \hat{\beta}_1^2 > \chi^2_{1,\alpha},$$

where $\chi^2_{1,\alpha}$ is the upper α-quantile of the χ^2_1 distribution.

$\boxed{2}$ V, W vector spaces.

(a) Let $V \subset W$ and suppose $\underset{\sim}{x} \in W^\perp$.

Now take any $\underset{\sim}{y} \in V$.

Since $V \subset W$ we have $\underset{\sim}{y} \in W$, so $\underset{\sim}{x}^T \underset{\sim}{y} = 0$.

This is true for all $\underset{\sim}{y} \in V$, so $\underset{\sim}{x} \in V^\perp$.

Therefore $W^\perp \subset V^\perp$.

(b) Let $\underset{\sim}{x} \in W$ and $\underset{\sim}{x} \in W^\perp$. Then $\underset{\sim}{x}^T \underset{\sim}{x} = 0$, giving $\underset{\sim}{x} = \underset{\sim}{0}$.

$\boxed{3}$ (a) Let $\underset{\sim}{x}$ be a nonzero eigenvector associated with λ.

Then $A\underset{\sim}{x} = \lambda \underset{\sim}{x}$, and by the idempotence of A we may write

$$AA\underset{\sim}{x} = A\underset{\sim}{x} = \lambda \underset{\sim}{x}$$

$$AA\underset{\sim}{x} = A\lambda\underset{\sim}{x} = \lambda^2 \underset{\sim}{x},$$

so that $\lambda \underset{\sim}{x} = \lambda^2 \underset{\sim}{x}$.

This can only hold if $\lambda \in \{0, 1\}$.

(b) By the spectral theorem we may write

$$A = PDP^T,$$

where $D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$, $\lambda_1, \dots, \lambda_n$ the eigenvalues of A.

Since $\lambda_1, \dots, \lambda_n \in \{0, 1\}$, we may discard all $\lambda_j = 0$.

Discard also the corresponding columns of P, leaving

$$A = P_s I_s P_s^T = P_s P_s^T, \qquad P_s^T P_s = I_s.$$

Set $G = P_s$.

So the columns of G are the eigenvectors of A corresponding to nonzero eigenvalues.

$\boxed{4}$ $Y_{ij} = \mu + A_i + \sigma_i \varepsilon_{ij}$, $i = 1, \dots, a$, $j = 1, \dots, n$, $\varepsilon_{ij} \overset{ind}{\sim} N(0,1)$, $A_i \overset{ind}{\sim} N(0, \sigma_A^2)$.

(a) $\underset{\sim}{Y} = \begin{bmatrix} \underset{\sim}{Y}_1 \\ \vdots \\ \underset{\sim}{Y}_a \end{bmatrix}$, $\underset{\sim}{Y}_i = \begin{bmatrix} Y_{i1} \\ \vdots \\ Y_{in} \end{bmatrix}$, $i = 1, \dots, a$

$$X = \begin{bmatrix} \underset{\sim}{1}_n \\ \vdots \\ \underset{\sim}{1}_n \end{bmatrix} = \underset{\sim}{1}_{na}, \quad Z = \begin{bmatrix} \underset{\sim}{1}_n & & \\ & \ddots & \\ & & \underset{\sim}{1}_n \end{bmatrix} = I_a \otimes \underset{\sim}{1}_n, \quad \underset{\sim}{u} = \begin{bmatrix} A_1 \\ \vdots \\ A_a \end{bmatrix}, \quad \underset{\sim}{e} = \begin{bmatrix} \varepsilon_{11} \\ \vdots \\ \varepsilon_{1n} \\ \vdots \\ \varepsilon_{a1} \\ \vdots \\ \varepsilon_{an} \end{bmatrix}$$

(b) $$\operatorname{Cov} \underset{\sim}{y} = Z\left[\sigma_A^2 I_a\right] Z^T + \begin{bmatrix} \sigma_1^2 I & & \\ & \ddots & \\ & & \sigma_a^2 I \end{bmatrix}$$

$$= \sigma_A^2 \begin{bmatrix} \underset{\sim}{1}_n & & \\ & \ddots & \\ & & \underset{\sim}{1}_n \end{bmatrix} \begin{bmatrix} \underset{\sim}{1}_n^T & & \\ & \ddots & \\ & & \underset{\sim}{1}_n^T \end{bmatrix} + \begin{pmatrix} \sigma_1^2 I_n & & \\ & \ddots & \\ & & \sigma_a^2 I_n \end{pmatrix}$$

$$= \begin{bmatrix} \sigma_A^2 J_n + \sigma_i^2 I_n & & \\ & \ddots & \\ & & \sigma_A^2 J_n + \sigma_a^2 I \end{bmatrix}.$$

$$=: V$$

(c) $$V^{-1} = \begin{bmatrix} \frac{1}{\sigma_1^2}\left(I_n - \frac{\sigma_A^2}{\sigma_1^2 + n\sigma_A^2} J_n\right) & & \\ & \ddots & \\ & & \frac{1}{\sigma_a^2}\left(I_n - \frac{\sigma_A^2}{\sigma_a^2 - n\sigma_A^2} J_n\right) \end{bmatrix}$$

(d) $$\hat{\mu}_{gls} = \left(X^T V^{-1} X\right)^{-1} X^T V^{-1} \underset{\sim}{y}.$$

$$X^T V^{-1} X = \begin{pmatrix} \underset{\sim}{1}_n \\ \vdots \\ \underset{\sim}{1}_n \end{pmatrix}^T \begin{pmatrix} \ddots & & \\ & \frac{1}{\sigma_i^2}\left(I_n - \frac{\sigma_A^2}{\sigma_i^2 + n\sigma_A^2} J_n\right) & \\ & & \ddots \end{pmatrix} \begin{pmatrix} \underset{\sim}{1}_n \\ \vdots \\ \underset{\sim}{1}_n \end{pmatrix}$$

$$= \sum_{i=1}^{a} \frac{1}{\sigma_i^2}\left(n - \frac{n^2\sigma_A^2}{\sigma_i^2 + n\sigma_A^2}\right)$$

$$= \sum_{i=1}^{a} \frac{n}{\sigma_i^2}\left(\frac{\sigma_i^2}{\sigma_i^2 + n\sigma_A^2}\right)$$

$$= \sum_{i=1}^{a} \frac{n}{\sigma_i^2 + n\sigma_A^2}.$$

$$X^T V^{-1} \underset{\sim}{y} = \begin{pmatrix} \underset{\sim}{1}_n \\ \vdots \\ \underset{\sim}{1}_n \end{pmatrix}^T \begin{pmatrix} \ddots & & \\ & \frac{1}{\sigma_i^2}\left(I_n - \frac{\sigma_A^2}{\sigma_i^2 + n\sigma_A^2} J_n\right) & \\ & & \ddots \end{pmatrix} \begin{bmatrix} \underset{\sim}{y}_1 \\ \vdots \\ \underset{\sim}{y}_a \end{bmatrix}$$

$$= \sum_{i=1}^{a} \frac{1}{\sigma_i^2}\left(\underset{\sim}{1}_n^T \underset{\sim}{y}_i - \frac{n\sigma_A^2}{\sigma_i^2 + n\sigma_A^2} \underset{\sim}{1}_n^T \underset{\sim}{y}_i\right)$$

$$= \sum_{i=1}^{a} \frac{n\bar{y}_{i\cdot}}{\sigma_i^2 + n\sigma_A^2}.$$

So

$$\hat{\mu}_{gls} = \frac{\sum_{i=1}^{a} \frac{n\bar{y}_{i\cdot}}{\sigma_i^2 + n\sigma_A^2}}{\sum_{i=1}^{a} \frac{n}{\sigma_i^2 + n\sigma_A^2}}.$$

(c) Let $\underset{\sim}{c} = [1]$ and $\underset{\sim}{d} = \underset{\sim}{e}_i$ Then

$$\hat{v}_i = \hat{\mu}_{gls} + \underset{\sim}{e}_i^T [\sigma_A^2 I_a] \begin{bmatrix} \underset{\sim}{1}_n^T & & \\ & \ddots & \\ & & \underset{\sim}{1}_n^T \end{bmatrix} \begin{bmatrix} \ddots & & \\ & \frac{1}{\sigma_i^2}\left(I_n - \frac{\sigma_A^2}{\sigma_i^2 + n\sigma_A^2} J_n\right) & \\ & & \ddots \end{bmatrix} \left[\begin{pmatrix} \underset{\sim}{y}_1 \\ \vdots \\ \underset{\sim}{y}_a \end{pmatrix} - \begin{pmatrix} \underset{\sim}{1}_n \hat{\mu}_{gls} \\ \vdots \\ \underset{\sim}{1}_n \hat{\mu}_{gls} \end{pmatrix}\right]$$

$$= \hat{\mu}_{gls} + \sigma_A^2 \left[\frac{1}{\sigma_i^2}\left(\underset{\sim}{1}_n^T (\underset{\sim}{y}_j - \underset{\sim}{1}_n \hat{\mu}_{gls}) - \frac{n\sigma_A^2}{\sigma_i^2 + n\sigma_A^2} \underset{\sim}{1}_n^T (\underset{\sim}{y}_j - \underset{\sim}{1}_n \hat{\mu}_{gls})\right)\right]$$

$$= \hat{\mu}_{gls} + \frac{n\sigma_A^2}{\sigma_i^2 + n\sigma_A^2}(\bar{y}_{i\cdot} - \hat{\mu}_{gls})$$

$$= \left(\frac{\sigma_i^2}{\sigma_i^2 + n\sigma_\delta^2}\right)\hat{\mu}_{gls} + \left(\frac{n\sigma_\delta^2}{\sigma_i^2 + n\sigma_\delta^2}\right)\overline{y}_{i\cdot}$$