STAT 714 hw 1

Matrix representation of linear model, the matrix inverse, solving $\mathbf{Ax} = \mathbf{b}$, linear independence

1. Four treatments will be compared in an experiment in which four subjects are assigned to each treatment according to a Latin Square design. There are two blocking variables—row and column blocking variables—with four levels each. The block and treatment arrangement will follow the diagram

	B_1	B_2	B_3	B_4
A_1	1	2	3	4
A_2	2	1	4	3
A_3	3	4	2	1
A_4	4	3	1	2

   where A_1, A_2, A_3, A_4 and B_1, B_2, B_3, B_4 are block effects and the numbers in the cell indicate what treatment is applied at the block combinations. The resulting data will be analyzed assuming the linear model

$$Y_{ijk} = \mu + A_i + B_j + \alpha_k + \varepsilon_{ijk}, \quad i, j, k \in \{1, 2, 3, 4\},$$

   where μ is a mean, $\alpha_1, ..., \alpha_4$ are fixed treatment effects, A_i are independent Normal $(0, \sigma_A^2)$, B_j are independent Normal $(0, \sigma_B^2)$, ε_{ijk} are independent Normal $(0, \sigma_\varepsilon^2)$, and the A_i, B_j, and ε_{ijk} are independent. Write the linear model in matrix notation $\mathbf{Y} = \mathbf{Xb} + \mathbf{Zu} + \boldsymbol{\varepsilon}$, where $\mathbf{b}$ contains fixed parameters and $\mathbf{u}$ contains random effects. Write out the entries of each vector and matrix.

2. For a matrix $\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}$ with $\mathbf{A}$ and $\mathbf{D}$ invertible, verify that

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{B}\mathbf{E}^{-1}\mathbf{C}\mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{B}\mathbf{E}^{-1} \\ -\mathbf{E}^{-1}\mathbf{C}\mathbf{A}^{-1} & \mathbf{E}^{-1} \end{bmatrix},$$

   where $\mathbf{E} = \mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}$.

3. Show that for $p \neq 0$ and $p + nq \neq 0$ we have

$$(p\mathbf{I}_n + q\mathbf{J}_n)^{-1} = \frac{1}{p}\Big(\mathbf{I}_n - \frac{q}{p + nq}\mathbf{J}_n\Big),$$

   where $\mathbf{I}_n$ is the $n \times n$ identity matrix and $\mathbf{J}_n = \mathbf{1}_n\mathbf{1}_n^T$ is an $n \times n$ matrix of ones.

4. Let $\mathbf{X}$ be an $n \times p$ matrix. Describe the change in $\mathbf{X}$ when is it is premultiplied by $\big(\mathbf{I}_n - \frac{1}{n}\mathbf{1}_n\mathbf{1}_n^T\big)$.

5. Characterize the solution set of $\mathbf{Ax} = \mathbf{b}$ (provided the system of equations is consistent), where

$$\mathbf{A} = \begin{bmatrix} 3 & 5 & -2 \\ -3 & -2 & -1 \\ 6 & 1 & 5 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} -7 \\ 1 \\ 4 \end{bmatrix}.$$

6. Show that if two nonzero vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal, then $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent.

7. Let $\{\mathbf{v}_1, \mathbf{v}_2\}$ be a set of linearly independent vectors in $\mathbb{R}^n$ and let

$$\mathbf{u}_1 = \mathbf{v}_1 \quad \text{and} \quad \mathbf{u}_2 = \mathbf{v}_2 - \left(\frac{\mathbf{v}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1.$$

Show that $\{\mathbf{u}_1, \mathbf{u}_2\}$ is linearly independent. *Hint: Show that* $\mathbf{u}_1$ *and* $\mathbf{u}_2$ *are orthogonal.*