STAT 714 hw 2

Dimension of a subspace, bases, rank, orthogonal complements, orthogonal projections

1. Let W be a subspace of $\mathbb{R}^n$ with an orthogonal basis $\{\mathbf{w}_1, \ldots, \mathbf{w}_p\}$ and let $\{\mathbf{v}_1, \ldots, \mathbf{v}_q\}$ be an orthogonal basis for $W^\perp$.
   (a) Show that the set $\{\mathbf{w}_1, \ldots, \mathbf{w}_p, \mathbf{v}_1, \ldots, \mathbf{v}_q\}$ is linearly independent.
   (b) State whether $\text{Span}\{\mathbf{w}_1, \ldots, \mathbf{w}_p, \mathbf{v}_1, \ldots, \mathbf{v}_q\} = \mathbb{R}^n$. Prove your statement.
   (c) Show that $p + q = n$.
   (d) Show whether the statement is true or not: For every $\mathbf{x} \in \mathbb{R}^n$, we have $\mathbf{x} \in W$ or $\mathbf{x} \in W^\perp$.
2. Let $\mathbf{U} = [\mathbf{u}_1, \ldots, \mathbf{u}_p]$, where $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is an orthonormal basis for a subspace W of $\mathbb{R}^n$. Show that the orthogonal projection of $\mathbf{y}$ onto W is given by $\hat{\mathbf{y}} = \mathbf{U}\mathbf{U}^T\mathbf{y}$.
3. Let $\mathbf{y} = (1, 1, 1)^T$ and let $\mathbf{v}_1 = (2, -5, 1)^T$ and $\mathbf{v}_2 = (4, -1, 2)^T$.
   (a) Produce an orthonormal basis for $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$.
   (b) Give the orthogonal projection $\hat{\mathbf{y}}$ of $\mathbf{y}$ onto $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$.
4. Show that if W and V are subspaces of $\mathbb{R}^n$ such that $W \subset V$, then $\dim W \leq \dim V$.
5. Let $\mathbf{A} = \sum_{k=1}^{r} \mathbf{u}_k \mathbf{v}_k^T$ for some vectors $\mathbf{u}_1, \ldots, \mathbf{u}_r \in \mathbb{R}^m$ and $\mathbf{v}_1, \ldots, \mathbf{v}_r \in \mathbb{R}^n$. Show that $\text{rank}\,\mathbf{A} \leq r$.
6. Consider the linear model given by

   $$Y_{ij} = \mu + \alpha_i + \beta_i x_{ij} + \varepsilon_{ij}, \quad i = 1, 2, \quad j = 1, 2, 3,$$

   with $x_{ij} = j$ for $i = 1, 2$ and $j = 1, 2, 3$, and where the ε_{ij} are $\text{Normal}(0, \sigma^2)$ random variables.
   (a) Put the model equations in matrix form $\mathbf{y} = \mathbf{X}\mathbf{b} + \boldsymbol{\varepsilon}$.
   (b) Give a basis for $\text{Col}\,\mathbf{X}$.
   (c) Give $\text{rank}\,\mathbf{X}$.
   (d) Give $\dim \text{Nul}\,\mathbf{X}$.
   (e) Give $\dim(\text{Col}\,\mathbf{X})^\perp$.
   (f) Give a basis for the orthogonal complement of $\text{Col}\,\mathbf{X}$.
   (g) Give the orthogonal projection of the vector $\mathbf{y} = (5, 6, 8, 4, 3, 1)^T$ onto $\text{Nul}\,\mathbf{X}^T$.
   (h) Give the orthogonal projection of the same $\mathbf{y}$ onto $\text{Col}\,\mathbf{X}$.
7. Let $\mathbf{y} \in \mathbb{R}^n$ and $\mathbf{X} \in \mathbb{R}^{n \times (n-1)}$ such that $\mathbf{1}^T\mathbf{y} = 0$ and $\mathbf{X}^T\mathbf{1} = \mathbf{0}$, where $\mathbf{1}$ is an $n \times 1$ vector of ones, and suppose $\mathbf{X}$ has full column rank. Show that $\mathbf{y} \in \text{Col}\,\mathbf{X}$.