STAT 714 hw 6

Cochran's theorem, ANOVA

Do problem 5.23 from Monahan. In addition:

1. Let V be a subspace in $\mathbb{R}^n$ and let $\mathbf{u} \in V^{\perp}$. Show that $\text{proj}_V \mathbf{u} = \mathbf{0}$.

2. Let $Z_1, \dots, Z_q \overset{\text{ind}}{\sim} \text{Normal}(0, 1)$ and let $\mu_1, \dots, \mu_q$ be real numbers. Derive the mean and variance of the random variable $W = \sum_{j=1}^{q}(Z_j + \mu_j)^2$.

3. Let $Y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \varepsilon_{ijk}$ for $i = 1, \dots, a$, $j = 1, \dots, b$ and $k = 1, \dots, n$, where $\varepsilon_{ijk} \overset{\text{ind}}{\sim} \text{Normal}(0, \sigma^2)$. Suppose the α_i and β_j represent *main effects* of factors A and B, which have a and b treatment levels, respectively, and the $(\alpha\beta)_{ij}$ represent *interaction effects* between A and B, which we will denote by AB. Note that the number of observations n is the same for all combinations of treatment levels; such a setup is called a *balanced design*.

   (a) Suppose $a = 2$ and $b = 2$.

      i. Give the design matrix $\mathbf{X}$ and the vector of parameters $\mathbf{b}$ in the matrix representation $\mathbf{y} = \mathbf{X}\mathbf{b} + \mathbf{e}$ of the model.

      ii. Let $\bar{\mu}_{i.} = (1/2)\sum_{j=1}^{2}(\mu + \alpha_i + \beta_j + (\alpha\beta)_{ij})$ for $i = 1, 2$ and $\bar{\mu}_{.j} = (1/2)\sum_{i=1}^{2}(\mu + \alpha_i + \beta_j + (\alpha\beta)_{ij})$ for $j = 1, 2$. Check whether these contrasts are estimable in the model $\mathbf{y} = \mathbf{X}\mathbf{b} + \mathbf{e}$.

      iii. Write down the matrix $\mathbf{C}$ such that $\mathbf{C}\mathbf{b} = \mathbf{0}$ imposes the constraints

$$\sum_{i=1}^{a}\alpha_i = 0, \quad \sum_{j=1}^{b}\beta_j = 0, \quad \text{and} \quad \sum_{i=1}^{a}(\alpha\beta)_{ij} = 0 \text{ for all } j \text{ and } \sum_{j=1}^{b}(\alpha\beta)_{ij} = 0 \text{ for all } i. \tag{1}$$

      iv. Give the matrix $\begin{bmatrix}\mathbf{X}^T\mathbf{X} \\ \mathbf{C}\end{bmatrix}$ and the vector $\begin{bmatrix}\mathbf{X}^T\mathbf{y} \\ \mathbf{0}\end{bmatrix}$.

      v. Under the constraint, give the least-squares estimators of all the parameters

$$\mu, \alpha_1, \alpha_2, \beta_1, \beta_2, (\alpha\beta)_{11}, (\alpha\beta)_{12}, (\alpha\beta)_{21}, (\alpha\beta)_{22}$$

      in terms of the response values Y_{ijk}.

      vi. Give the least-squares estimators of the contrasts $\bar{\mu}_{1.}$, $\bar{\mu}_{2.}$, $\bar{\mu}_{.1}$, and $\bar{\mu}_{.2}$.

      vii. Give the vector $\mathbf{P}_{\mathbf{X}}\mathbf{y}$ in terms of the values Y_{ijk}.

      viii. Make a complete ANOVA table with expressions for sums of squares in terms of the responses Y_{ijk} and give the degrees of freedom. Use the sequential sum of squares idea based on Cochran's theorem. Optionally, give the noncentrality parameters in terms of the model parameters μ, α_i, and β_j. Create your table in this form (like the table on pg 115 of [2]):

Source	SS	df	ϕ (optional)
Mean			
A			
B			
AB			
Error			

ix. Give $\hat{\sigma}^2$ in terms of the response values Y_{ijk}.

(b) Now give the ANOVA table for any $a \geq 2$ and $b \geq 2$ (you do not need to work this out step-by-step; you may just "extrapolate" from your work in the first part).

(c) Use the data in the table, which is scanned from [1]. Let factor A be the "Compaction Method" and factor B be the "Aggregate Type".

Table 6.3 Tensile strength (psi) of asphaltic concrete specimens for two aggregate types with each of four compaction methods

	Compaction Method			
		Kneading		
Aggregate Type	*Static*	*Regular*	*Low*	*Very Low*
Basalt	68	126	93	56
	63	128	101	59
	65	133	98	57
Silicious	71	107	63	40
	66	110	60	41
	66	116	59	44

Source: A. M. Al-Marshed (1981), Compaction effects on asphaltic concrete durability. M.S. thesis, Civil Engineering, University of Arizona.

For the following you may use R, but you may NOT use any built-in functions for fitting linear models! If you use R, provide your code.

i. Obtain the values of the least-squares estimators of μ, the α_i, the β_j, and the $(\alpha\beta)_{ij}$ under the constraints in (1).

ii. Give the sums of squares corresponding to the mean, factor A, factor B, the interaction AB, and the error term. Give the degrees of freedom corresponding to each sum of squares.

iii. Give $\hat{\sigma}^2$.

(d) Now consider the same data set with some observations removed so that the design is *unbalanced*.

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Without using any built-in linear models functions in R, obtain the values of the least-squares estimators of μ, the α_i, the β_j, and the $(\alpha\beta)_{ij}$ under the constraints

$$\sum_{i=1}^{a} n_{i.}\alpha_i = 0, \quad \sum_{j=1}^{b} n_{.j}\beta_j = 0, \quad \text{and} \quad \sum_{i=1}^{a} n_{ij}(\alpha\beta)_{ij} = 0 \ \forall\ j \text{ and } \sum_{j=1}^{b} n_{ij}(\alpha\beta)_{ij} = 0 \ \forall\ i.$$

References

[1] R. O. Kuehl. *Design of Experiments: Statistical Principles of Research Design and Analysis.* Duxbury/Thomson Learning, 2000. Google-Books-ID: mIV2QgAACAAJ.

[2] John F Monahan. *A primer on linear models.* CRC Press, 2008.