

STAT 714 hw 7

Likelihood ratio test (F test) for general linear hypothesis

1. Let $Y_{ijk} = \mu_{ij} + \varepsilon_{ijk}$, $\varepsilon_{ijk} \stackrel{\text{ind}}{\sim} \text{Normal}(0, \sigma^2)$ for $i = 1, \dots, a$ and $j = 1, \dots, b$, $k = 1, \dots, n_{ij}$. In the model, μ_{ij} represents the mean response of experimental units under treatment level i of factor A and treatment level j of factor B , for $i = 1, \dots, a$ and $j = 1, \dots, b$. This is called a two-way factorial design.
 - (a) Write the model in matrix form $\mathbf{y} = \mathbf{X}\mathbf{b} + \mathbf{e}$.
 - (b) Assume $a = b = 2$, so that each factor has only two treatment levels. Consider testing the hypothesis $H_0: \mu_{ik} - \mu_{jk} = \mu_{im} - \mu_{jm}$ for all i, j, k, m .
 - i. Give an interpretation of the null hypothesis.
 - ii. Give H_0 in the form $H_0: \mathbf{K}^T \mathbf{b} = \mathbf{m}$.
 - iii. Let $n_{11} = 5$, $n_{12} = 3$, $n_{21} = 5$, and $n_{22} = 4$ and suppose $\sigma = 1/3$. Give the power of the likelihood ratio test of H_0 when $\mu_{11} = 1$, $\mu_{12} = 2$, $\mu_{21} = 1$, and $\mu_{22} = 3$. Use significance level $\alpha = 0.05$.
 - iv. Suppose one has not yet collected data, but one wants to know what number of replicates in each group will be necessary to achieve a certain statistical power. Use R to generate a plot showing the power of the likelihood ratio test of H_0 against the value of the signal-to-noise ratio $\text{SNR} = \|\mathbf{K}^T \mathbf{b}\|^2 / \sigma^2$. Include power curves under $n = 3, 4, 5, 6, 7, 8, 9$, where n is the number of replicates at each treatment level combination (so use $n_{ij} = n$ for all i, j).
 - v. Suppose $\mu_{11} = 1$, $\mu_{12} = 2$, $\mu_{21} = 1$, and $\mu_{22} = 3$ and $\sigma = 1/3$. Use your plot to determine the necessary number of replicates per treatment group to reject H_0 with probability at least 0.90 when testing at the $\alpha = 0.05$ significance level.
 - (c) To test for the significance of a *main effect* of factor A , one tests $H_0: \bar{\mu}_{i.} = \bar{\mu}_{j.}$ for all i, j , where $\bar{\mu}_{i.} = b^{-1} \sum_{k=1}^b \mu_{ik}$ for each $i = 1, \dots, a$. The null hypothesis for testing significance of a main effect of factor B is formulated analogously. For this part suppose $a = 3$ and $b = 2$. In answering the following, it may be helpful to draw a table like this one for yourself:

$$\begin{array}{cc}
 \mu_{11} & \mu_{12} \\
 \mu_{21} & \mu_{22} \\
 \mu_{31} & \mu_{32}
 \end{array}$$

For each of the following, give the matrix $\mathbf{K}$ such that we may formulate the hypothesis of interest as $H_0: \mathbf{K}^T \mathbf{b} = \mathbf{0}$.

- i. For testing the significance of the main effect of treatment A .
 - ii. For testing the significance of the main effect of treatment B .
 - iii. For testing the significance of an interaction between factors A and B . In the absence of interaction, the differences in means across the levels of one factor do not depend on the level of the other factor.
- (d) Use the data in the image below scanned from [1].

Table 6.19 Tensile strength (psi) of asphaltic concrete specimens for two aggregate types with each of three kneading compaction methods

Aggregate Type	Compaction Method			Aggregate Means ($\bar{y}_{i..}$)
	Kneading			
	Regular	Low	Very Low	
Basalt	106 108	93 101 98	56	
Means ($\bar{y}_{1j.}$)	107.0	97.3	56	93.7
Silicious	107 110 116	63 60	40 41 44	
Means ($\bar{y}_{2j.}$)	111.0	61.5	41.7	72.6
Compaction means ($\bar{y}_{.j.}$)	109.4	83.0	45.3	

Fill out the ANOVA table without using any built-in linear models functions in R.

Source	SS	df	MS	F	p val
Total	(i)	(ii)			
Aggregate	(iii)	(iv)	(v)	(vi)	(vii)
Compaction	(viii)	(ix)	(x)	(xi)	(xii)
Interaction	(xiii)	(xiv)	(xv)	(xvi)	(xvii)
Error	(xviii)	(xix)	(xx)		

- This is $\mathbf{y}^T(\mathbf{I} - \mathbf{P}_1)\mathbf{y}$, where $\mathbf{P}_1$ is the orthogonal projection onto $\text{Span}\{\mathbf{1}_n\}$.
- This the the rank of $\mathbf{I} - \mathbf{P}_1$.
- This is the sum of squares for testing the main effect of the aggregate type, which is the value of

$$(\mathbf{K}^T \hat{\mathbf{b}} - \mathbf{m})^T [\mathbf{K}^T (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{K}]^{-1} (\mathbf{K}^T \hat{\mathbf{b}} - \mathbf{m}),$$

where $\mathbf{K}$ is the matrix such that $H_0: \mathbf{K}^T \mathbf{b} = \mathbf{m}$.

- Degrees of freedom corresponding to the main effect of the aggregate type.
- The is the sum of squares divided by the degrees of freedom.
- The LRT test statistic for testing significance of the main effect of the aggregate type.
- The p-value of the LRT test of significance of the main effect of the aggregate type.
- This is the sum of squares for testing the main effect of the compaction method.
- Degrees of freedom corresponding to the main effect of the compaction method.
- The is the sum of squares divided by the degrees of freedom.
- The LRT test statistic for testing significance of the main effect of the compaction method.
- The p-value of the LRT test of significance of the main effect of the compaction method.
- This is the sum of squares for testing for an interaction.

- xiv. Degrees of freedom corresponding to the interaction.
- xv. The is the sum of squares divided by the degrees of freedom.
- xvi. The LRT test statistic for testing significance of the interaction.
- xvii. The p-value of the LRT test of significance of the interaction.
- xviii. This is $\mathbf{y}^T(\mathbf{I} - \mathbf{P}_{\mathbf{X}})\mathbf{y}$.
- xix. The rank of the matrix $\mathbf{I} - \mathbf{P}_{\mathbf{X}}$.
- xx. The sum of squares divided by the degrees of freedom.

2. Let $Y_i = \beta_1 x_{1i} + \dots + \beta_p x_{pi} + \varepsilon_i$, $\varepsilon_i \stackrel{\text{ind}}{\sim} \text{Normal}(0, \sigma^2)$ for $i = 1, \dots, n$. Assume the matrix $\mathbf{X} = (x_{ij})_{1 \leq i \leq n, 1 \leq j \leq p}$ has rank p .

(a) Show that the size- α likelihood ratio test of $H_0: \beta_j = 0$ versus $H_1: \beta_j \neq 0$ is

$$\text{Reject } H_0 \text{ if } \sqrt{n} \hat{\Omega}_{jj}^{-1/2} |\hat{\beta}_j| / \hat{\sigma} > t_{n-p, \alpha/2},$$

where $\hat{\Omega}_{jj}$ is entry j on the diagonal of $\hat{\Omega} = (n^{-1} \mathbf{X}^T \mathbf{X})^{-1}$.

(b) Show that $\sqrt{n} \hat{\Omega}_{jj}^{-1/2} \hat{\beta}_j / \hat{\sigma} \sim t_{n-p}(\phi = \sqrt{n} \hat{\Omega}_{jj}^{-1/2} \beta_j / \sigma)$.

(c) Show that the noncentrality parameter $\phi = \sqrt{n} \hat{\Omega}_{jj}^{-1/2} \beta_j / \sigma$ can be written as

$$\phi = \frac{\beta_j}{\sigma} \|(\mathbf{I}_n - \mathbf{P}_{\mathbf{X}_{-j}}) \mathbf{x}_j\|_2,$$

where $\mathbf{X}_{-j}$ is the matrix $\mathbf{X}$ with column j removed and $\mathbf{x}_j$ is column j of $\mathbf{X}$.

(d) Set $n = 100$, $\sigma = 1$ and, for $p = 20, 40, 80, 90$, generate an $n \times p$ design matrix $\mathbf{X}$ having rows from the $\text{Normal}(\mathbf{0}, \mathbf{I}_n)$ distribution. Then plot the power curves of the test in part (a) at size 0.05 for testing $H_0: \beta_1 = 0$ versus $H_1: \beta_1 \neq 0$. Put the four power curves on the same plot.

(e) Describe the effect of having large p on the power of the test.

3. Let $\mathbf{y} = \mathbf{X}\mathbf{b} + \mathbf{e}$ and let $\mathbf{K}$ be a $p \times s$ matrix with columns in $\text{Col } \mathbf{X}^T$ and $\mathbf{m}$ be an $s \times 1$ vector. Let $\tilde{\mathbf{K}}$ and $\tilde{\mathbf{m}}$ be any other matrix and vector such that

$$\{\mathbf{b} : \mathbf{K}^T \mathbf{b} = \mathbf{m}\} = \{\mathbf{b} : \tilde{\mathbf{K}}^T \mathbf{b} = \tilde{\mathbf{m}}\}.$$

Show that the value of the F-statistic is the same regardless of whether one specifies the null hypothesis as $H_0: \mathbf{K}^T \mathbf{b} = \mathbf{m}$ or as $H_0: \tilde{\mathbf{K}}^T \mathbf{b} = \tilde{\mathbf{m}}$.

References

- [1] R. O. Kuehl. *Design of Experiments: Statistical Principles of Research Design and Analysis*. Duxbury/Thomson Learning, 2000. Google-Books-ID: mIV2QgAACAAJ.