STAT 714 hw 7

Likelihood ratio test (F test) for general linear hypothesis

1. Let $Y_{ijk} = \mu_{ij} + \varepsilon_{ijk}$, $\varepsilon_{ijk} \overset{\text{ind}}{\sim} \text{Normal}(0, \sigma^2)$ for $i = 1, \dots, a$ and $j = 1, \dots, b$, $k = 1, \dots, n_{ij}$. In the model, μ_{ij} represents the mean response of experimenal units under treatment level i of factor A and treatment level j of factor B, for $i = 1, \dots, a$ and $j = 1, \dots, b$. This is called a two-way factorial design.

   (a) Write the model in matrix form $\mathbf{y} = \mathbf{X}\mathbf{b} + \mathbf{e}$.

   (b) Assume $a = b = 2$, so that each factor has only two treatment levels. Consider testing the hypothesis H_0: $\mu_{ik} - \mu_{jk} = \mu_{im} - \mu_{jm}$ for all i, j, k, m.

      i. Give an interpretation of the null hypothesis.

      The null hypothesis states that there is "no interaction" between the two factors; that is, the affect on the response mean of one factor does not depend on the level of the other factor.

      ii. Give H_0 in the form H_0: $\mathbf{K}^T\mathbf{b} = \mathbf{m}$.

      Let $\mathbf{m} = \mathbf{0}$ and set $\mathbf{K}^T = [1 \quad -1 \quad -1 \quad 1]$ or $\mathbf{K}^T = [-1 \quad 1 \quad 1 \quad -1]$, or any scalar multiple of this.

      iii. Let $n_{11} = 5$, $n_{12} = 3$, $n_{21} = 5$, and $n_{22} = 4$ and suppose $\sigma = 1/3$. Give the power of the likelihood ratio test of H_0 when $\mu_{11} = 1$, $\mu_{12} = 2$, $\mu_{21} = 1$, and $\mu_{22} = 3$. Use significance level $\alpha = 0.05$.

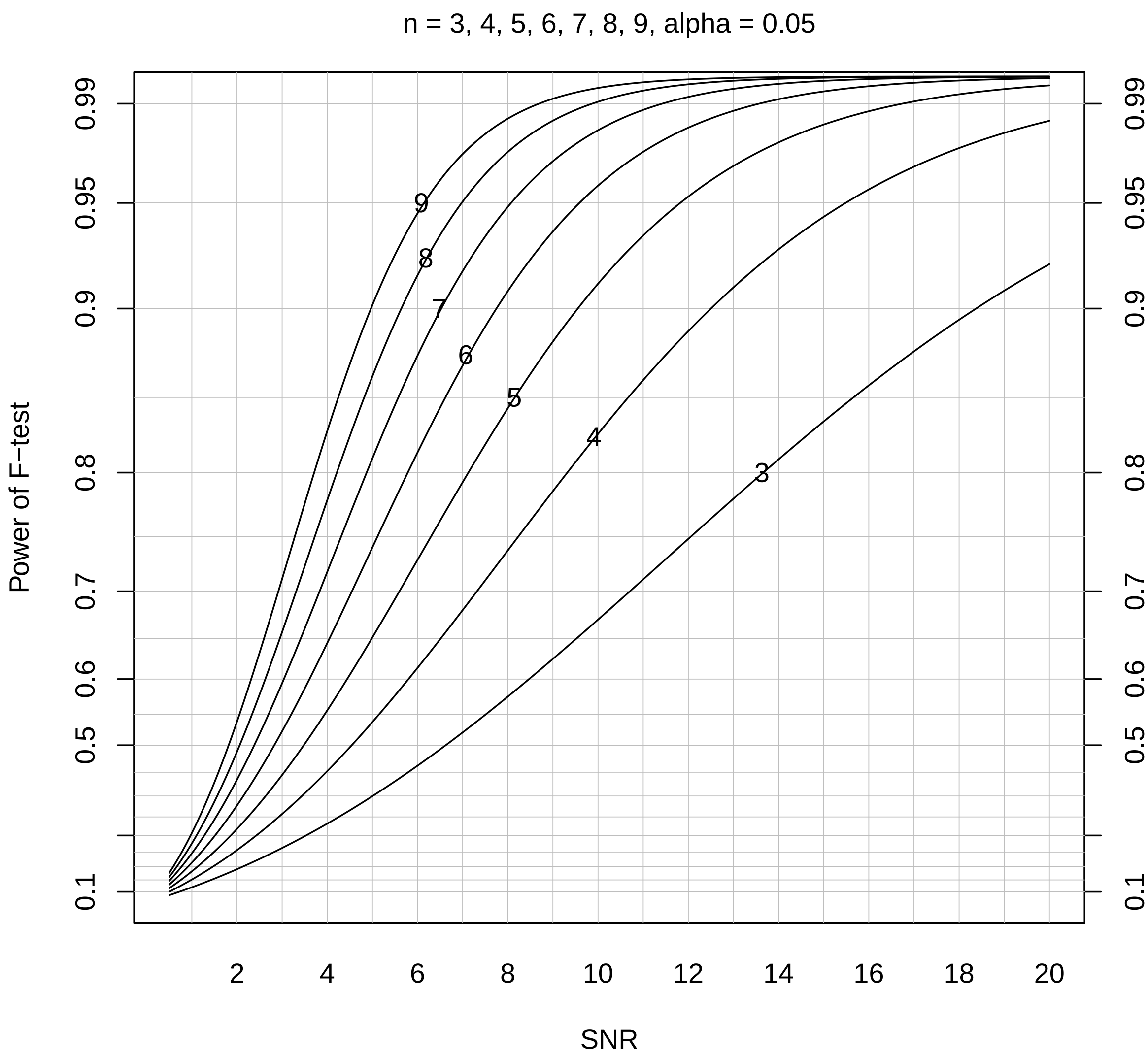
```r
nn <- c(5,3,5,4)
mu <- c(1,2,1,3)
sigma <- 1/3

a <- 2
b <- 2
N <- sum(nn)

# build X
X <- matrix(0,N,a*b)
m <- 1
for(i in 1:a)
  for(j in 1:b){

    k <- b*(i-1) + j
    X[m:(m + nn[k] - 1),k] <- rep(1,nn[k])
    m <- m + nn[k]

  }

# generate y
e <- rnorm(N,0,sigma)
y <- as.numeric(X %*% mu) + e

# construct K
K <- c(1,-1,-1,1)

# compute noncentrality parameter
Hinv <- solve( t(K) %*% solve(t(X) %*% X ) %*% K)
ncp <- as.numeric(t(t(K) %*% mu) %*% Hinv %*% t(K) %*% mu / sigma^2)

# compute power
alpha <- 0.05
powF <- 1 - pf(qf(1 - alpha,df1=1, df2=N-4),df1=1, df2=N-4, ncp=ncp)
powF
## [1] 0.7984592
```

iv. Suppose one has not yet collected data, but one wants to know what number of replicates in each group will be necessary to achieve a certain statistical power. Use R to generate a plot showing the power of the likelihood ratio test of H_0 against the value of the signal-to-noise ratio SNR $= \|\mathbf{K}^T\mathbf{b}\|^2/\sigma^2$. Include power curves under $n = 3, 4, 5, 6, 7, 8, 9$, where n is the number of replicates at each treatment level combination (so use $n_{ij} = n$ for all i, j).

Note that in the balanced design (equal replications in each treatment group), we have $\mathbf{X}^T\mathbf{X} = n_N$. Moreover, $\mathbf{K}^T\mathbf{K} = 4\mathbf{I}_2$, so we have $[\mathbf{K}^T(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{K}]^{-1} = n/4$. This, together with $\mathbf{m} = \mathbf{0}$, gives the noncentrality parameter

$$\phi = \frac{1}{\sigma^2}(\mathbf{K}^T\mathbf{b} - \mathbf{m})^T[\mathbf{K}^T(\mathbf{X}^T\mathbf{X})^{-}\mathbf{K}]^{-1}(\mathbf{K}^T\mathbf{b} - \mathbf{m}) = \frac{n}{4\sigma^2}\|\mathbf{K}^T\mathbf{b}\|^2 = \frac{n}{4}\,\text{SNR}\,.$$

```
nn <- c(3,4,5,6,7,8,9)
snr <- seq(1/2,20,length=200)
powF <- matrix(NA,length(nn),length(snr))
for(i in 1:length(nn)){

  n <- nn[i]
  ncp <- n * snr/4
  powF[i,] <- 1-pf(qf(1-alpha,df1=1,df2=n*4-4),df1=1,df2=n*4-4,ncp=ncp)
}

plot(NA, xlim = range(snr), ylim = exp(exp(c(.1,.99))),
     yaxt = "n", xaxt = "n", ylab = "Power of F-test",xlab = "SNR")
at <- c(.1,.3,.5,.6,.7,.8,.9,.95,.99)
axis(side = 2, at = exp(exp(at)), labels = at)
axis(side = 4, at = exp(exp(at)), labels = at)
abline(h = exp(exp(c(seq(.1,.95, by = .05),.99))),lwd = .5,col = "gray")
axis(side = 1, at = seq(2,20, by = 2), tick = FALSE)
abline(v = 1:20, lwd = .5, col = "gray")
pow_at <- seq(.8,.95,length = length(nn))
for(i in 1:length(nn)){
  lines(exp(exp(powF[i,])) ~ snr)
  snr_pow <- sum(exp(exp(powF[i,])) < exp(exp(pow_at[i])))
  text(x = snr[snr_pow], y = exp(exp(pow_at[i])), label = nn[i])
}
mtext(side = 3, text = paste("n = ",paste(nn,collapse=", "),
                     ", alpha = ",alpha,sep = ""), line = 1)
```



v. Suppose $\mu_{11} = 1$, $\mu_{12} = 2$, $\mu_{21} = 1$, and $\mu_{22} = 3$ and $\sigma = 1/3$. Use your plot to determine the necessary number of replicates per treatment group to reject H_0 with probability at least 0.90 when testing at the $\alpha = 0.05$ significance level.

```
sigma <- 1/3
mu <- c(1,2,1,3)
snr <- sum( (t(K)%*% mu)^2)/sigma^2
snr
## [1] 9
```

These settings give a signal to noise ratio of 9. According to the plot we would need

6 replicates per treatment group in order to detect an interaction at the $\alpha = 0.05$ significance level.

(c) To test for the significance of a *main effect* of factor A, one tests H_0: $\bar{\mu}_{i.} = \bar{\mu}_{j.}$ for all i, j, where $\bar{\mu}_{i.} = b^{-1} \sum_{k=1}^{b} \mu_{ik}$ for each $i = 1, \ldots, a$. The null hypothesis for testing significance of a main effect of factor B is formulated analogously. For this part suppose $a = 3$ and $b = 2$. In answering the following, it may be helpful to draw a table like this one for yourself:

$$\begin{matrix} \mu_{11} & \mu_{12} \\ \mu_{21} & \mu_{22} \\ \mu_{31} & \mu_{32} \end{matrix}$$

For each of the following, give the matrix $\mathbf{K}$ such that we may formulate the hypothesis of interest as H_0: $\mathbf{K}^T\mathbf{b} = \mathbf{0}$.

i. For testing the significance of the main effect of treatment A.

We wish to test

$$H_0\colon\ (\mu_{11} + \mu_{12})/2 = (\mu_{21} + \mu_{22})/2 \text{ and } (\mu_{21} + \mu_{22})/2 = (\mu_{31} + \mu_{32})/2.$$

We can reformulate this as

$$H_0\colon\ \mu_{11} + \mu_{12} - \mu_{21} - \mu_{22} = 0 \text{ and } \mu_{21} + \mu_{22} - \mu_{31} - \mu_{32} = 0.$$

With $\mathbf{b} = [\mu_{11}\ \mu_{12}\ \mu_{21}\ \mu_{22}\ \mu_{31}\ \mu_{32}]^T$, we see that we can express the hypothesis as H_0: $\mathbf{K}^T\mathbf{b} = \mathbf{0}$, where

$$\mathbf{K}^T = \begin{bmatrix} 1 & 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{bmatrix}.$$

ii. For testing the significance of the main effect of treatment B.

We wish to test

$$H_0\colon\ (\mu_{11} + \mu_{21} + \mu_{31})/3 = (\mu_{12} + \mu_{22} + \mu_{32})/3.$$

We can reformulate this as

$$H_0\colon\ \mu_{11} + \mu_{21} + \mu_{31} - \mu_{12} - \mu_{22} - \mu_{32} = 0.$$

With $\mathbf{b} = [\mu_{11}\ \mu_{12}\ \mu_{21}\ \mu_{22}\ \mu_{31}\ \mu_{32}]^T$, we see that we can express the hypothesis as H_0: $\mathbf{K}^T\mathbf{b} = \mathbf{0}$, where

$$\mathbf{K}^T = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix}.$$

iii. For testing the significance of an interaction between factors A and B. In the absence of interaction, the differences in means across the levels of one factor do not depend on the level of the other factor.

We wish to test

$$H_0\colon\ \mu_{11} - \mu_{12} = \mu_{21} - \mu_{22} \text{ and } \mu_{21} - \mu_{22} = \mu_{31} - \mu_{32}.$$

Other combinations of i, j, k, m are redundant. We can reformulate the above as

$$H_0\colon\ \mu_{11} - \mu_{12} - \mu_{21} + \mu_{22} = 0 \text{ and } \mu_{21} - \mu_{22} - \mu_{31} + \mu_{32} = 0.$$

With $\mathbf{b} = [\mu_{11}\ \mu_{12}\ \mu_{21}\ \mu_{22}\ \mu_{31}\ \mu_{32}]^T$, we see that we can express the hypothesis as H_0: $\mathbf{K}^T\mathbf{b} = \mathbf{0}$, where

$$\mathbf{K}^T = \begin{bmatrix} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{bmatrix}.$$

(d) Use the data in the image below scanned from [1].

Table 6.19 Tensile strength (psi) of asphaltic concrete specimens for two aggregate types with each of three kneading compaction methods

	Compaction Method			
	Kneading			
Aggregate Type	*Regular*	*Low*	*Very Low*	*Aggregate Means* ($\bar{y}_{i..}$)
Basalt	106	93	56	
	108	101		
		98		
Means ($\bar{y}_{1j.}$)	107.0	97.3	56	93.7
Silicious	107	63	40	
	110	60	41	
	116		44	
Means ($\bar{y}_{2j.}$)	111.0	61.5	41.7	72.6
Compaction means ($\bar{y}_{.j.}$)	109.4	83.0	45.3	

Fill out the ANOVA table without using any built-in linear models functions in R.

Source	SS	df	MS	F	p val
Total	(i)	(ii)			
Aggregate	(iii)	(iv)	(v)	(vi)	(vii)
Compaction	(viii)	(ix)	(x)	(xi)	(xii)
Interaction	(xiii)	(xiv)	(xv)	(xvi)	(xvii)
Error	(xviii)	(xix)	(xx)		

i. This is $\mathbf{y}^T(\mathbf{I} - \mathbf{P_1})\mathbf{y}$, where $\mathbf{P_1}$ is the orthogonal projection onto Span$\{\mathbf{1}_n\}$.

ii. This the the rank of $\mathbf{I} - \mathbf{P_1}$.

iii. This is the sum of squares for testing the main effect of the aggregate type, which is the value of

$$(\mathbf{K}^T\hat{\mathbf{b}} - \mathbf{m})^T[\mathbf{K}^T(\mathbf{X}^T\mathbf{X})^-\mathbf{K}]^{-1}(\mathbf{K}^T\hat{\mathbf{b}} - \mathbf{m}),$$

where $\mathbf{K}$ is the matrix such that H_0: $\mathbf{K}^T\mathbf{b} = \mathbf{m}$.

iv. Degrees of freedom corresponding to the main effect of the aggregate type.

v. The is the sum of squares divided by the degrees of freedom.

vi. The LRT test statistic for testing significance of the main effect of the aggregate type.

vii. The p-value of the LRT test of significance of the main effect of the aggregate type.

viii. This is the sum of squares for testing the main effect of the compaction method.

ix. Degrees of freedom corresponding to the main effect of the compaction method.

x. The is the sum of squares divided by the degrees of freedom.

xi. The LRT test statistic for testing significance of the main effect of the compaction method.

xii. The p-value of the LRT test of significance of the main effect of the compaction method.

xiii. This is the sum of squares for testing for an interaction.

xiv. Degrees of freedom corresponding to the interaction.

xv. The is the sum of squares divided by the degrees of freedom.

xvi. The LRT test statistic for testing significance of the interaction.

xvii. The p-value of the LRT test of significance of the interaction.

xviii. This is $\mathbf{y}^T(\mathbf{I} - \mathbf{P_X})\mathbf{y}$.

xix. The rank of the matrix $\mathbf{I} - \mathbf{P_X}$.

xx. The sum of squares divided by the degrees of freedom.

```r
y <- c(106, 108, 107, 110, 116, 93, 101, 98, 63, 60, 56, 40, 41, 44)
nn <- c(2,3,3,2,1,3);a <- 3;b <- 2;N <- sum(nn)

# build X
X <- matrix(0,N,a*b)
m <- 1
for(i in 1:a)
  for(j in 1:b){
k <- b*(i-1) + j
X[m:(m + nn[k] - 1),k] <- rep(1,nn[k])
m <- m + nn[k]
  }

bhat <- solve(t(X) %*% X) %*% t(X) %*% y

# compute the sums of squares:
SST <- sum( (y - mean(y))^2)

KA <- t(rbind(c(1,1,-1,-1,0,0),c(0,0,1,1,-1,-1)))
HinvA <- solve( t(KA) %*% solve(t(X) %*% X) %*% KA )
SSA <- t( t(KA) %*% bhat) %*% HinvA %*% (t(KA) %*% bhat)

KB <- c(1,-1,1,-1,1,-1)
HinvB <- solve( t(KB) %*% solve(t(X) %*% X) %*% KB )
SSB <- t( t(KB) %*% bhat) %*% HinvB %*% (t(KB) %*% bhat)

KAB <- t(rbind(c(1,-1,-1,1,0,0),
           c(0,0,1,-1,-1,1)))
HinvAB <- solve( t(KAB) %*% solve(t(X) %*% X) %*% KAB )
SSAB <- t( t(KAB) %*% bhat) %*% HinvAB %*% (t(KAB) %*% bhat)

SSE <- sum( (y -  X %*% bhat)^2 )

MSA <- SSA / 2
MSB <- SSB / 1
MSAB <- SSAB / 2
MSE <- SSE / ( N - a*b)

F_A <- MSA / MSE
F_B <- MSB / MSE
F_AB <- MSAB / MSE

qf(.999,df1=2,8)
## [1] 18.49365
qf(.999,df1=1,8)
## [1] 25.41476
```

Source	SS	df	MS	F	p val
Total	10963.21	13			
Aggregate	710.4537	1	710.4537	63.2686	< 0.001
Compaction	6806.452	2	3403.226	303.0702	< 0.001
Interaction	953.4492	2	476.7246	42.45414	< 0.001
Error	89.83333	8	11.22917		

2. Let $Y_i = \beta_1 x_{1i} + \dots \beta_p x_{pi} + \varepsilon_i$, $\varepsilon_i \overset{\text{ind}}{\sim} \text{Normal}(0, \sigma^2)$ for $i = 1, \dots, n$. Assume the matrix $\mathbf{X} = (x_{ij})_{1 \le i \le n, 1 \le j \le p}$ has rank p.

   (a) Show that the size-α likelihood ratio test of H_0: $\beta_j = 0$ versus H_1: $\beta_j \neq 0$ is

   $$\text{Reject } H_0 \text{ if } \sqrt{n}\hat{\Omega}_{jj}^{-1/2}|\hat{\beta}_j|/\hat{\sigma} > t_{n-p,\alpha/2},$$

   where $\hat{\Omega}_{jj}$ is entry j on the diagonal of $\hat{\mathbf{\Omega}} = (n^{-1}\mathbf{X}^T\mathbf{X})^{-1}$.

   Choose $\mathbf{K}^T = \mathbf{e}_j^T$, where $\mathbf{e}_j$ is the $p \times 1$ vector with every entry equal to zero except for entry j, which is equal to 1. Then one can show that the F statistic for testing H_0: $\mathbf{K}^T\mathbf{b} = \mathbf{0}$ is equal to $n\hat{\Omega}_{jj}^{-1}(\hat{\beta}_j)^2/\hat{\sigma}^2$. The size-$\alpha$ LRT rejects H_0 when this is greater than $F_{1,n-p,\alpha}$. Since $T \sim t_{n-p} \implies T^2 \sim F_{1,n-p}$, we have $(t_{n-p,\alpha/2})^2 = F_{1,n-p,\alpha}$, so an equivalent decision rule is $\sqrt{n}\hat{\Omega}_{jj}^{-1/2}|\hat{\beta}_j|/\hat{\sigma} > t_{n-p,\alpha/2}$

   (b) Show that $\sqrt{n}\hat{\Omega}_{jj}^{-1/2}\hat{\beta}_j/\hat{\sigma} \sim t_{n-p}(\phi = \sqrt{n}\hat{\Omega}_{jj}^{-1/2}\beta_j/\sigma)$.

   We have

   $$\begin{aligned} \frac{\sqrt{n}\hat{\Omega}_{jj}^{-1/2}\hat{\beta}_j}{\hat{\sigma}} &= \frac{\sqrt{n}\hat{\Omega}_{jj}^{-1/2}\hat{\beta}_j/\sigma}{\sqrt{((n-p)\hat{\sigma}^2/\sigma^2)/(n-p)}} \\ &= \frac{\sqrt{n}\hat{\Omega}_{jj}^{-1/2}(\hat{\beta}_j - \beta_j)/\sigma + \sqrt{n}\hat{\Omega}_{jj}^{-1/2}\beta_j/\sigma}{\sqrt{((n-p)\hat{\sigma}^2/\sigma^2)/(n-p)}} \\ &= \frac{Z + \phi}{\sqrt{W/(n-p)}}, \end{aligned}$$

   where $Z \sim \text{Normal}(0,1)$, $W \sim \chi^2_{n-p}$, and $\phi = \sqrt{n}\hat{\Omega}_{jj}^{-1/2}\beta_j/\sigma$.

   (c) Show that the noncentrality parameter $\phi = \sqrt{n}\hat{\Omega}_{jj}^{-1/2}\beta_j/\sigma$ can be written as

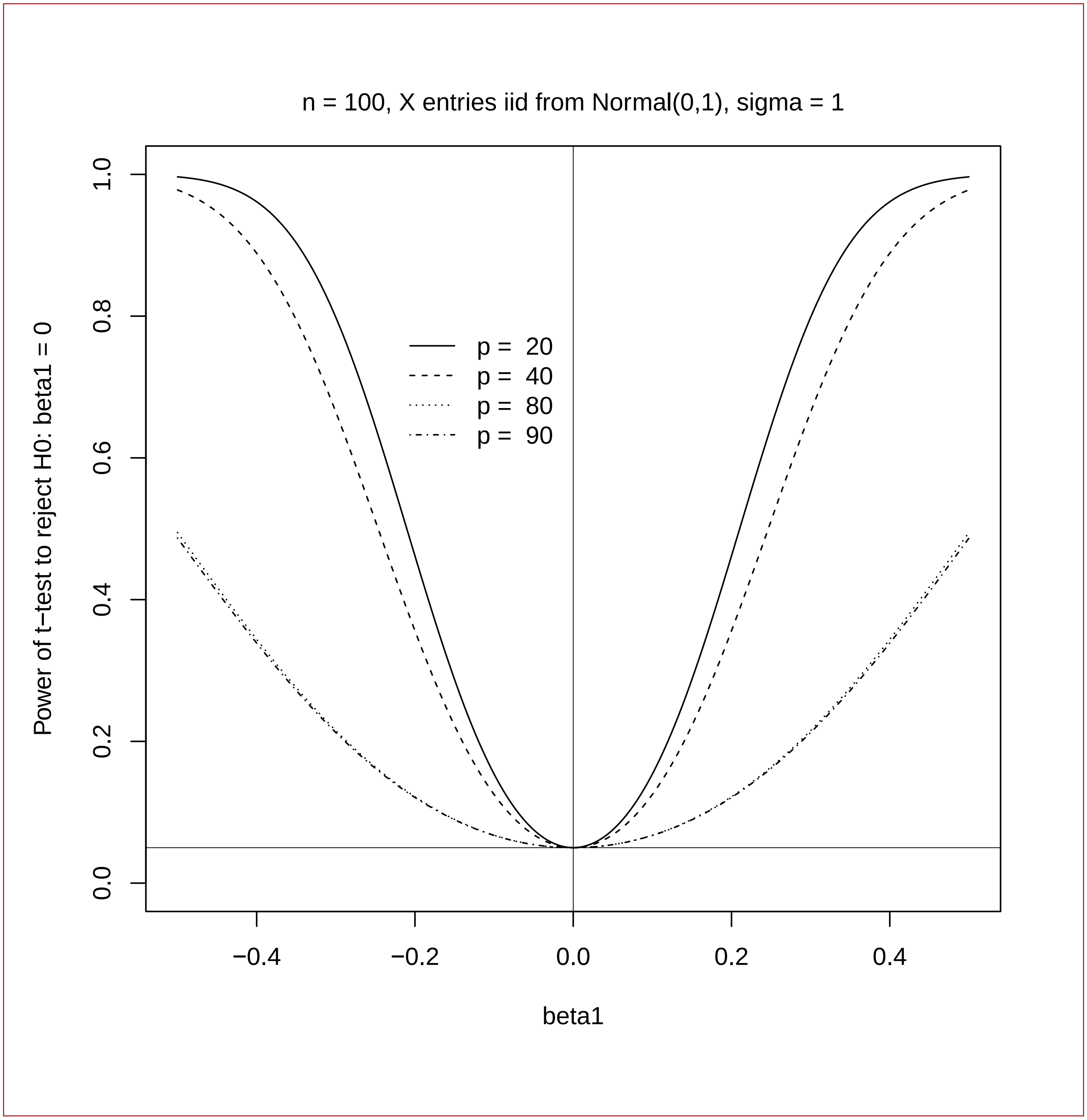
   $$\phi = \frac{\beta_j}{\sigma}\|(\mathbf{I}_n - \mathbf{P}_{\mathbf{X}_{-j}})\mathbf{x}_j\|_2,$$

   where $\mathbf{X}_{-j}$ is the matrix $\mathbf{X}$ with column j removed and $\mathbf{x}_j$ is column j of $\mathbf{X}$.

For convenience, set $j = 1$, and then partition $\mathbf{X}$ as $\mathbf{X} = [\mathbf{x}_1 \ \mathbf{X}_{-1}]$. Then write $\mathbf{X}^T\mathbf{X}$ as a block matrix. Use the block inverse formula to obtain the $(1,1)$ entry of $(\mathbf{X}^T\mathbf{X})^{-1}$ as $(\mathbf{x}_1^T\mathbf{x}_1 - \mathbf{x}_1^T\mathbf{X}_{-1}^T(\mathbf{X}_{-1}^T\mathbf{X}_{-1})^{-1}\mathbf{X}_{-1}^T\mathbf{x}_1)^{-1} = (\mathbf{x}_1^T(\mathbf{I}_n - \mathbf{P}_{\mathbf{X}_{-1}})\mathbf{x}_1)^{-1}$. The answer follows.

(d) Set $n = 100$, $\sigma = 1$ and, for $p = 20, 40, 80, 90$, generate an $n \times p$ design matrix $\mathbf{X}$ having rows from the Normal$(\mathbf{0}, \mathbf{I}_n)$ distribution. Then plot the power of the test in part (a) as a function of the true value of the parameter β_j at size 0.05 for testing H_0: $\beta_1 = 0$ versus H_1: $\beta_1 \neq 0$. Put the four power curves on the same plot.

```r
#### The t-test power thingy:
rm(list=ls())

n <- 100
pp <- c(20,40,80,90)
alpha <- 0.05
sigma <- 1
beta1 <- seq(-1/2,1/2,length = 200)
pow_mat <- matrix(0,nrow = length(beta1),length(pp))
for( j in 1:length(pp)){

  p <- pp[j]
  X <- matrix(rnorm(n*p),n,p)
  Omega <- solve( t(X)%*%X ) * n
  Omega11 <- Omega[1,1]

  ncp <- sqrt(n) * abs(beta1) / (sigma * sqrt(Omega11) )
  t_crit <- qt(1-alpha/2,df = n - p)
  pow_mat[,j] <- 1-(pt(t_crit,df=n-p,ncp=ncp)-pt(-t_crit,df=n-p,ncp=ncp))

}


plot(NA,
 xlim = range(beta1),
 ylim = c(0,1),
 xlab = "beta1",
 ylab = "Power of t-test to reject H0: beta1 = 0")
for(j in 1:length(pp)) lines(pow_mat[,j] ~ beta1, lty = j)
abline(h = alpha, lwd = 1/2)
abline(v = 0, lwd = 1/2)
legend( x = grconvertX(from = "nfc", to = "user", .35),
    y = .8,
    legend = paste("p = ",pp),
    lty = 1:length(pp),
    bty = "n")
mtext(side = 3,
  text = paste("n = ",n,
               ", X entries iid from Normal(0,1), sigma = ",sigma,
               sep=""),
  line = 1)
```



(e) Describe the effect of having large p on the power of the test.

3. Let $\mathbf{y} = \mathbf{X}\mathbf{b} + \mathbf{e}$ and let $\mathbf{K}$ be a $p \times s$ matrix with columns in $\operatorname{Col} \mathbf{X}^T$ and $\mathbf{m}$ be an $s \times 1$ vector. Let $\tilde{\mathbf{K}}$ and $\tilde{\mathbf{m}}$ be any other matrix and vector such that

$$\{\mathbf{b} : \mathbf{K}^T\mathbf{b} = \mathbf{m}\} = \{\mathbf{b} : \tilde{\mathbf{K}}^T\mathbf{b} = \tilde{\mathbf{m}}\}.$$

Show that the value of the F-statistic is the same regardless of whether one specifies the null hypothesis as H_0: $\mathbf{K}^T\mathbf{b} = \mathbf{m}$ or as H_0: $\tilde{\mathbf{K}}^T\mathbf{b} = \tilde{\mathbf{m}}$.

See book page 134.

References

[1] R. O. Kuehl. *Design of Experiments: Statistical Principles of Research Design and Analysis*. Duxbury/Thomson Learning, 2000. Google-Books-ID: mIV2QgAACAAJ.