STAT 714 fa 2025

Linear algebra review 1/6

Vectors and matrices, matrix inverse

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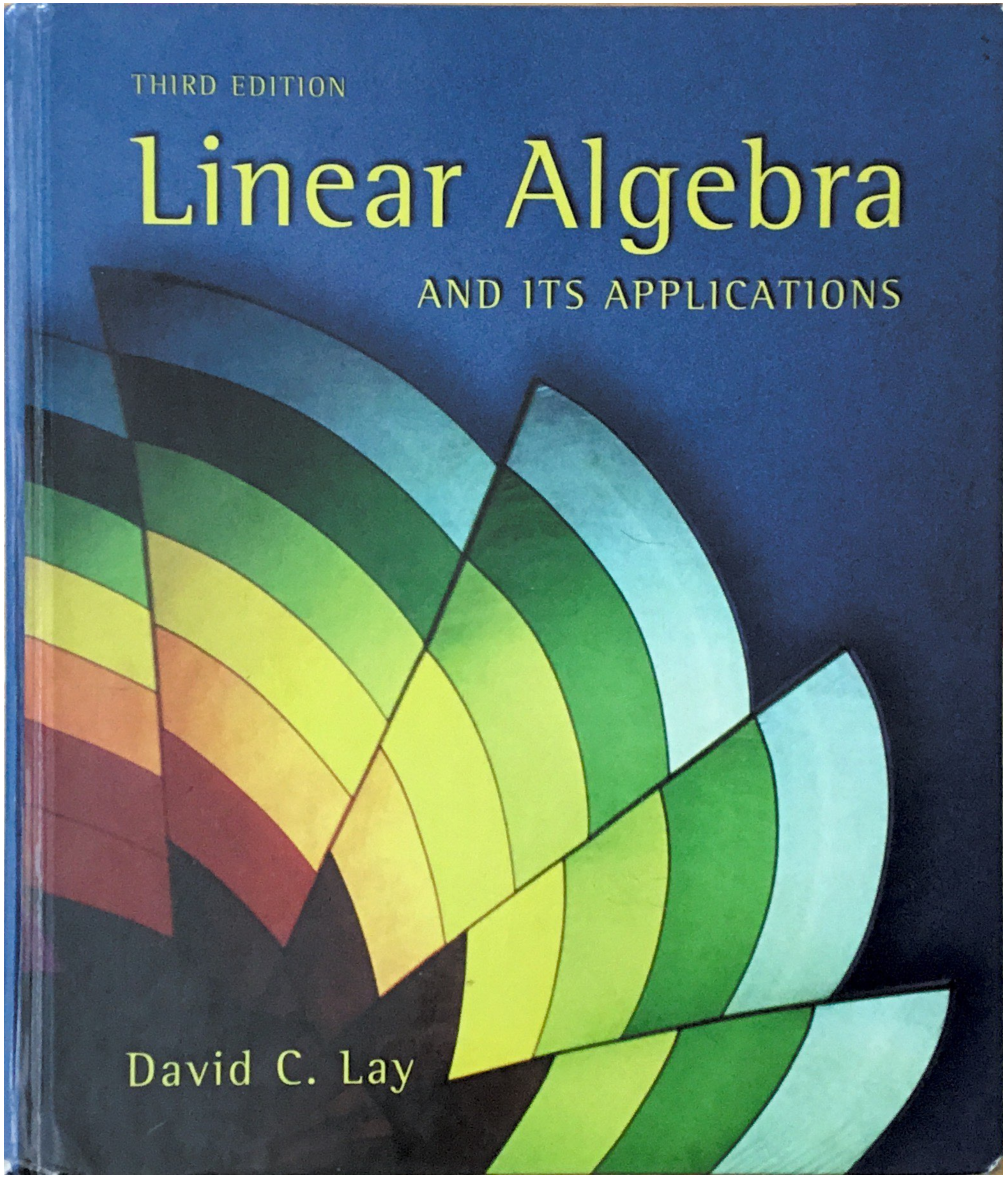
These slides are an instructional aid; their sole purpose is to display, during the lecture, definitions, plots, results, etc. which take too much time to write by hand on the blackboard. They are not intended to explain or expound on any material.

These notes include scanned excerpts from Lay (2003):



1. Vectors in $\mathbb{R}^n$
2. Matrices in $\mathbb{R}^{m \times n}$
3. Inverse of a matrix

A *vector* $\mathbf{x} \in \mathbb{R}^n$ is an $n \times 1$ column matrix of real numbers

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

Sums and scalar multiples of vectors

Given $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $c \in \mathbb{R}$, the sum $\mathbf{x} + \mathbf{y}$ and the scalar multiple of $\mathbf{x}$ by c are

$$\mathbf{x} + \mathbf{y} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix} \quad \text{and} \quad c\mathbf{x} = \begin{bmatrix} cx_1 \\ cx_2 \\ \vdots \\ cx_n \end{bmatrix}.$$

No surprises here:

ALGEBRAIC PROPERTIES OF $\mathbb{R}^n$

For all $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ in $\mathbb{R}^n$ and all scalars c and d:

(i) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

(ii) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

(iii) $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$

(iv) $\mathbf{u} + (-\mathbf{u}) = -\mathbf{u} + \mathbf{u} = \mathbf{0}$, where $-\mathbf{u}$ denotes $(-1)\mathbf{u}$

(v) $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$

(vi) $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$

(vii) $c(d\mathbf{u}) = (cd)(\mathbf{u})$

(viii) $1\mathbf{u} = \mathbf{u}$

Inner product of vectors

The *inner product* of $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ is defined as $\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + \cdots + u_n v_n$.

"dot" product

No surprises here either:

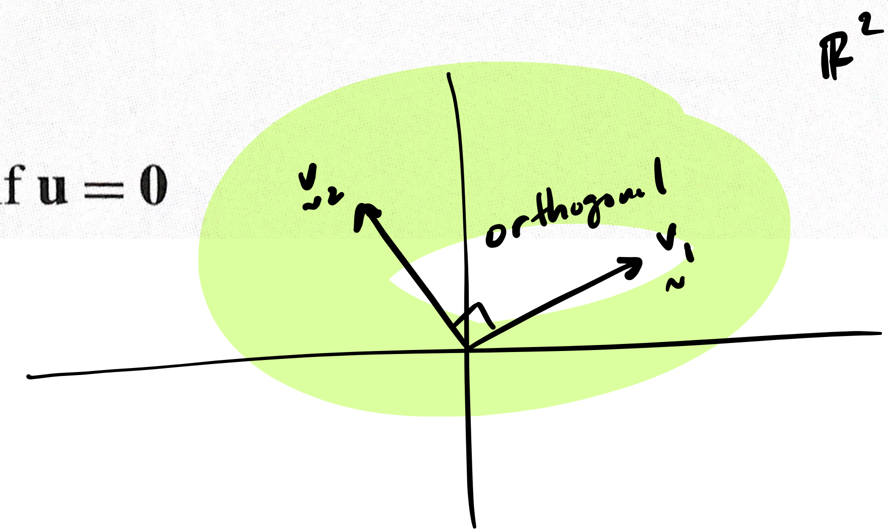
Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in $\mathbb{R}^n$, and let c be a scalar. Then

a. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

b. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$

c. $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$

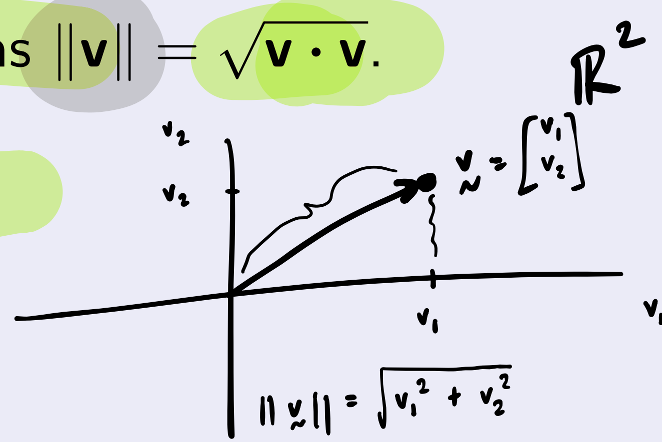
d. $\mathbf{u} \cdot \mathbf{u} \geq 0$, and $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$



Length or Euclidean norm of a vector

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$.

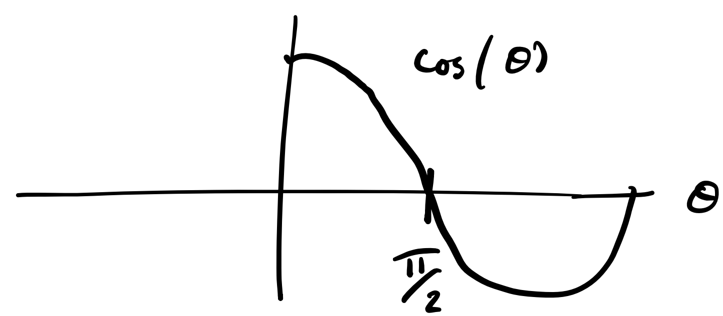
1. The *length* or *Euclidean norm* of $\mathbf{v}$ is defined as $\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$.
2. We call $\mathbf{v}$ a *unit vector* if $\|\mathbf{v}\| = 1$.
3. We say $\mathbf{u}$ and $\mathbf{v}$ are *orthogonal* if $\mathbf{u} \cdot \mathbf{v} = 0$.
4. The *distance* between $\mathbf{v}$ and $\mathbf{u}$ is $\|\mathbf{v} - \mathbf{u}\|$.
5. The *angle* between $\mathbf{v}$ and $\mathbf{u}$ is $\cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}\right)$.



Exercises: Let

$$\mathbf{u} = \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} \sqrt{3}/2 \\ -1/2 \end{bmatrix}.$$

1. Which pairs of vectors are orthogonal?
2. Which vectors are unit vectors?



Pythagorean theorem, Cauchy-Schwarz and Triangle inequalities.

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$.

1. *Pythagorean theorem*: $\mathbf{u}$ and $\mathbf{v}$ are orthogonal iff $\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$.
2. *Cauchy-Schwarz inequality*: $|\mathbf{u}\cdot\mathbf{v}| \leq \|\mathbf{u}\|\|\mathbf{v}\|$
3. *Triangle inequality*: $\|\mathbf{u}+\mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$

Prove the results.

(1) $\|\underset{\sim}{u} + \underset{\sim}{v}\|^2 = (\underset{\sim}{u} + \underset{\sim}{v})\cdot(\underset{\sim}{u} + \underset{\sim}{v})$

$= \sum_{i=1}^{n} (u_i + v_i)^2$

$= \sum_{i=1}^{n} (u_i^2 + v_i^2 + 2u_i v_i)$

$$= \sum_{i=1}^{n} u_i^2 + \sum_{i=1}^{n} v_i^2 + 2\sum_{i=1}^{n} u_i v_i$$

$$= \|\underset{\sim}{u}\|^2 + \|\underset{\sim}{v}\|^2 + 2\,\underset{\sim}{u}\cdot\underset{\sim}{v}$$

$$\|\underset{\sim}{u}+\underset{\sim}{v}\|^2 = \|\underset{\sim}{u}\|^2 + \|\underset{\sim}{v}\|^2 \iff \underset{\sim}{u}\cdot\underset{\sim}{v} = 0.$$

($\underset{\sim}{u}$, $\underset{\sim}{v}$, orthogonal).

② Cauchy-Schwarz

claim: $|\underset{\sim}{u}\cdot\underset{\sim}{v}| \leq \|\underset{\sim}{u}\|\,\|\underset{\sim}{v}\|$

proof: $\|\underset{\sim}{u}+\underset{\sim}{v}\|^2 \geq 0$

$$\Rightarrow \|\underset{\sim}{u}+\underset{\sim}{v}\|^2 = \|\underset{\sim}{u}\|^2 + \|\underset{\sim}{v}\|^2 + 2\,\underset{\sim}{u}\cdot\underset{\sim}{v} \geq 0$$

$$\Rightarrow -\underset{\sim}{u}\cdot\underset{\sim}{v} \leq \frac{1}{2}\left(\|\underset{\sim}{u}\|^2 + \|\underset{\sim}{v}\|^2\right)$$

$$\underset{\sim}{u}\cdot\underset{\sim}{v} \geq -\frac{1}{2}\left(\|u\|^2 + \|\underset{\sim}{v}\|^2\right)$$

$\|\underset{\sim}{u}-\underset{\sim}{v}\|^2 \geq 0$

$$\Rightarrow \|\underset{\sim}{u}-\underset{\sim}{v}\|^2 = \|\underset{\sim}{u}\|^2 + \|\underset{\sim}{v}\|^2 - 2\,\underset{\sim}{u}\cdot\underset{\sim}{v} \geq 0$$

$$\Rightarrow \underset{\sim}{u}\cdot\underset{\sim}{v} \leq \frac{1}{2}\left(\|\underset{\sim}{u}\|^2 + \|\underset{\sim}{v}\|^2\right)$$

$$|\underset{\sim}{u}\cdot\underset{\sim}{v}| \le \frac{1}{2}\left(\|\underset{\sim}{u}\|^2 + \|\underset{\sim}{v}\|^2\right)$$

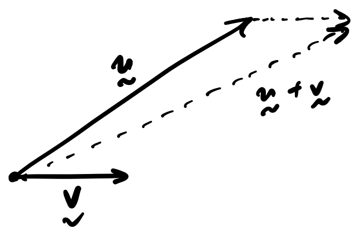
$$\underset{\sim}{a} = \underset{\sim}{u}\frac{1}{\|\underset{\sim}{u}\|} \qquad \underset{\sim}{b} = \underset{\sim}{v}\frac{1}{\|\underset{\sim}{v}\|}.$$

$$\|\underset{\sim}{a}\| = \sqrt{\underset{\sim}{a}\cdot\underset{\sim}{a}} = \sqrt{\frac{\underset{\sim}{u}\cdot\underset{\sim}{u}}{\|\underset{\sim}{u}\|\,\|\underset{\sim}{u}\|}} = 1$$

$$|\underset{\sim}{a}\cdot\underset{\sim}{b}| \le \frac{1}{2}\left(\underbrace{\|\underset{\sim}{a}\|^2}_{=1} + \underbrace{\|\underset{\sim}{b}\|^2}_{=1}\right) = 1$$

$$\Rightarrow \left|\frac{\underset{\sim}{u}\cdot\underset{\sim}{v}}{\|\underset{\sim}{u}\|\,\|\underset{\sim}{v}\|}\right| \le 1$$

$$\Rightarrow |\underset{\sim}{u}\cdot\underset{\sim}{v}| \le \|\underset{\sim}{u}\|\,\|\underset{\sim}{v}\|. \qquad \square$$



③ Triangle inequality

claim $\|u + v\| \le \|\underset{\sim}{u}\| + \|\underset{\sim}{v}\|$

proof:

$$\|\underset{\sim}{u}+\underset{\sim}{v}\|^2 = \|\underset{\sim}{u}\|^2 + \|\underset{\sim}{v}\|^2 + 2\,\underset{\sim}{u}\cdot\underset{\sim}{v}$$

$$\overset{C-S}{\le} \|\underset{\sim}{u}\|^2 + \|\underset{\sim}{v}\|^2 + 2\|\underset{\sim}{u}\|\,\|\underset{\sim}{v}\|$$

$$= \left(\|\underset{\sim}{u}\| + \|\underset{\sim}{v}\|\right)^2$$

$$\Rightarrow \|\underset{\sim}{u}+\underset{\sim}{v}\| \le \|\underset{\sim}{u}\| + \|\underset{\sim}{v}\| \qquad \square$$

$\underset{\sim}{u} \cdot \underset{\sim}{v} = 0$ "orthogonal"

Orthogonal and orthonormal sets of vectors

Let $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ be a set of vectors in $\mathbb{R}$.

1. We call $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ an *orthogonal set* of vectors if $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ for all $i \neq j$.
2. If in addition $\|\mathbf{v}_i\| = 1$ for $i = 1, \dots, n$, we call it an *orthonormal set*.

↑ unit vector

Example: The *elementary vectors*

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \quad \dots \quad, \quad \mathbf{e}_{n-1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

in $\mathbb{R}^n$ make an orthonormal set of vectors.

Linear combination

Given vectors $\mathbf{v}_1, \dots, \mathbf{v}_p \in \mathbb{R}^n$ and scalars $c_1, \dots, c_p \in \mathbb{R}$, the vector

$$\mathbf{y} = c_1 \mathbf{v}_1 + \cdots + c_p \mathbf{v}_p$$

is a *linear combination* of $\mathbf{v}_1, \dots, \mathbf{v}_p$ with weights $c_1, \dots, c_p$.

Example: We often decompose a vector as a linear combination of vectors, e.g.

$$\begin{bmatrix} 3 \\ 4 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

A matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ is a table of numbers $\underset{m\times n}{\mathbf{A}} = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix}$.

Sum of two matrices

Given $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{m\times n}$ and $c \in \mathbb{R}$, $\mathbf{A} + \mathbf{B}$ and the scalar multiple of $\mathbf{A}$ by c are

a single real number.

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} a_{11} + b_{11} & \dots & a_{1n} + b_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & \dots & a_{mn} + b_{mn} \end{bmatrix} \quad \text{and} \quad c\mathbf{A} = \begin{bmatrix} ca_{11} & \dots & ca_{1n} \\ \vdots & \ddots & \vdots \\ ca_{m1} & \dots & ca_{mn} \end{bmatrix}.$$

Extract rows, columns, or entries of a matrix $\mathbf{A}$ with

$$\text{row}_i(\mathbf{A}) = [a_{i1} \; \dots \; a_{in}], \quad \text{col}_j(\mathbf{A}) = \begin{bmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{bmatrix}, \quad (\mathbf{A})_{ij} = a_{ij}.$$

Again no surprises:

Let A, B, and C be matrices of the same size, and let r and s be scalars.

a. $A + B = B + A$

b. $(A + B) + C = A + (B + C)$

c. $A + 0 = A$

d. $r(A + B) = rA + rB$

e. $(r + s)A = rA + sA$

f. $r(sA) = (rs)A$

$\underset{m\times n}{A}\ \underset{n\times 1}{x}$ $\qquad \underset{m\times n}{A} = [\underset{m\times 1}{a_1} \cdots \underset{m\times 1}{a_n}]$

Product of a matrix and a vector

If $\mathbf{A}$ is an $m \times n$ matrix with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$ and $\mathbf{x} \in \mathbb{R}^n$, then

$$\underset{m\times 1}{\mathbf{A}\mathbf{x}} = x_1 \underset{m\times 1}{\mathbf{a}_1} + \cdots + x_n \underset{m\times 1}{\mathbf{a}_n}.$$

That is, $\mathbf{A}\mathbf{x}$ is a linear combination of the columns of $\mathbf{A}$ with weights from $\mathbf{x}$.

So $(\mathbf{A}\mathbf{x})_i = \text{row}_i(\mathbf{A})\mathbf{x} = \sum_{j=1}^n a_{ij}x_j$, where $(\mathbf{A}\mathbf{x})_i$ denotes entry i of $\mathbf{A}\mathbf{x}$.

Exercise: Give $\mathbf{A}\mathbf{x}$, where

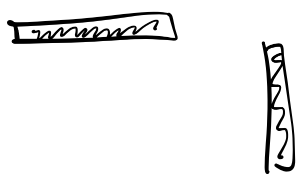
$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 2 \\ 1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}.$$

$$Ax = 1\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (-1)\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}$$

$$A_x = x_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

$$\left(A_x\right)_i = x_1 a_{i1} + \cdots + x_n a_{in}$$

$$= \text{row}_i(A)\ \underset{\sim}{x}$$



$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 2 \\ 1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}.$$

$$A\underset{\sim}{x} = \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}$$

$$\mathbf{I}_n = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{e}_1 & \cdots & \mathbf{e}_n \end{bmatrix}$$

Identity matrix

For each integer $n \geq 1$, the $n \times n$ identity matrix $\mathbf{I}_n$ is the $n \times n$ matrix with diagonal entries equal to 1 and all other entries equal to 0.

So $\mathbf{I}_n = [\mathbf{e}_1 \ \dots \ \mathbf{e}_n]$, where $\mathbf{e}_1, \dots, \mathbf{e}_n$ are the elementary basis vectors.

Exercise: For any $\mathbf{x} \in \mathbb{R}^n$, show that $\mathbf{I}_n\mathbf{x} = \mathbf{x}$.

$$\mathbf{I}_n \mathbf{x} = [\mathbf{e}_1 \cdots \mathbf{e}_n]\mathbf{x} = x_1 \mathbf{e}_1 + \dots + x_n \mathbf{e}_n$$

$$= x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \dots + x_n \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \mathbf{x}$$

$$\underbrace{\underset{m\times n}{A}\,\underset{n\times 1}{\tilde{b}_1}}_{m\times 1} = b_{11}\,\underset{m\times 1}{\tilde{a}_1} + \cdots + b_{1n}\,\underset{m\times 1}{\tilde{a}_n}$$

$$\underset{m\times n}{A}\,\underset{n\times p}{B} = \underset{m\times n}{A}\left[\underset{n\times 1}{\tilde{b}_1}\ \cdots\ \underset{n\times 1}{\tilde{b}_p}\right] = \underbrace{\left[\underset{m\times 1}{A\tilde{b}_1}\ \cdots\ \underset{m\times 1}{A\tilde{b}_p}\right]}_{m\times p}$$

must match

$m \times p$

Product of two matrices

If $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{B}$ is an $n \times p$ matrix with columns $\mathbf{b}_1, \ldots, \mathbf{b}_p$, then the product $\mathbf{AB}$ is the $m \times p$ matrix with columns $\mathbf{Ab}_1, \ldots, \mathbf{Ab}_p$.

Above is the definition of $\mathbf{AB}$. Below are some helper rules one can derive.

Theorem (Row-column, column-row rules for matrix multiplication)

If $\mathbf{A}$ is $m \times n$ and $\mathbf{B}$ is $n \times p$, then we have the two rules

1. *Row-column:* $(\mathbf{AB})_{ij} = \text{row}_i(\mathbf{A})\,\text{col}_j(\mathbf{B}) = \sum_{k=1}^{n} a_{ik} b_{kj}$.
2. *Column-row:* $\mathbf{AB} = \text{col}_1(\mathbf{A})\,\text{row}_1(\mathbf{B}) + \cdots + \text{col}_n(\mathbf{A})\,\text{row}_n(\mathbf{B})$.

Exercise: Give the matrix product $\mathbf{AB}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 2 \\ 1 & 1 & 0 \end{bmatrix}.$$

$$AB = A\begin{bmatrix} \underset{\sim}{b}_1 & \cdots & \underset{\sim}{b}_p \end{bmatrix} = \begin{bmatrix} A\underset{\sim}{b}_1 & \cdots & A\underset{\sim}{b}_p \end{bmatrix}$$

$$\Big(\underset{m\times n}{A}\ \underset{n\times p}{B}\Big)_{ij} = \Big(\begin{bmatrix} A\underset{\sim}{b}_1 & \cdots & A\underset{\sim}{b}_p \end{bmatrix}\Big)_{ij}$$

$$= \Big(\underset{m\times n}{A}\ \underset{n\times 1}{\underset{\sim}{b}_j}\Big)_i$$

$$= \Big(b_{1j}\underset{m\times 1}{\underset{\sim}{a}_1} + \cdots + b_{nj}\underset{m\times 1}{\underset{\sim}{a}_n} \Big)_i$$

$$= b_{1j}a_{i1} + \cdots + b_{nj}a_{in}$$

$$= \text{row}_i(A)\ \text{col}_j(B)$$

$$\underset{2\times 3}{\mathbf{A}} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix}, \quad \underset{3\times 3}{\mathbf{B}} = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 2 \\ 1 & 1 & 0 \end{bmatrix}.$$

$$\underset{2\times 3}{A}\;\underset{3\times 3}{B} = \begin{bmatrix} 5 & 4 & 2 \\ 3 & 3 & 0 \end{bmatrix}$$

2×3

~~$\underset{3\times 3}{B}\;\underset{2\times 3}{A}$~~

More unsurprising facts:

AB not same as BA.

If A is an $m \times n$ matrix, $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$, and c is a scalar, then:

a. $A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v}$;

b. $A(c\mathbf{u}) = c(A\mathbf{u})$.

Let A be an $m \times n$ matrix, and let B and C have sizes for which the indicated sums and products are defined.

a. $A(BC) = (AB)C$ (associative law of multiplication)

b. $A(B+C) = AB + AC$ (left distributive law)

c. $(B+C)A = BA + CA$ (right distributive law)

d. $r(AB) = (rA)B = A(rB)$ for any scalar r

e. $I_m A = A = A I_n$ (identity for matrix multiplication)

Transpose of a matrix

The *transpose* of an $m \times n$ matrix $\mathbf{A}$, denoted $\mathbf{A}^T$, is the $n \times m$ matrix of which the rows are the columns of $\mathbf{A}$.

One little surprise...

$$\underset{m\times n}{A} = [\underset{\sim}{a}_1 \cdots \underset{\sim}{a}_n] \qquad \underset{n\times m}{A^T} = \begin{bmatrix} \underset{\sim}{a}_1^T \\ \vdots \\ \underset{\sim}{a}_n^T \end{bmatrix}$$

Let A and B denote matrices whose sizes are appropriate for the following sums and products.

a. $(A^T)^T = A$

b. $(A + B)^T = A^T + B^T$

c. For any scalar r, $(rA)^T = rA^T$

d. $(AB)^T = B^T A^T$

Prove result d.

$$(AB)_{ij} = ((AB)^T)_{ji}$$

$$(B^T A^T)_{ji} = \text{row}_j(B^T)\ \text{col}_i(A^T)$$

A ($m \times n$) B ($n \times p$)

$$= [\text{col}_j(B)]^T [\text{row}_i(A)]^T$$

$$= b_{1j} a_{i1} \cdots b_{nj} a_{in}$$

$$= \text{row}_i(A)\ \text{col}_j(B)$$

$$= \sum_{k=1}^{n} b_{kj} a_{ik}$$

$$= (AB)_{ij}$$

$$\Rightarrow (AB)^T = B^T A^T.$$

$$(ABC)^T = (A(BC))^T = (BC)^T A^T = C^T B^T A^T$$

Inner and outer products with the transpose

Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors in $\mathbb{R}^n$.

1. We can write the inner product of $\mathbf{u}$ and $\mathbf{v}$ as $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T\mathbf{v}$.
2. The outer product of $\mathbf{u}$ and $\mathbf{v}$ is defined as the $n \times n$ matrix $\mathbf{u}\mathbf{v}^T$.

Exercise:

1. Compute inner and outer product of $\mathbf{u} = (1, 2, 3)^T$ and $\mathbf{v} = (1, 0, -1)^T$.
2. Let $\mathbf{X} = [\mathbf{x}_1 \dots \mathbf{x}_n]^T$ be an $n \times p$ matrix. Give $\mathbf{X}^T\mathbf{X}$.

$X = [\underset{\sim}{x}_1 \cdots \underset{\sim}{x}_n]^T$

Multiplication of partitioned matrices

Partitioned matrices can be multiplied with the row-column rule as though the block entries were scalars.

Exercise: Find **AB**, where these are the partitioned matrices

$$A = \left[\begin{array}{ccc|cc} 2 & -3 & 1 & 0 & -4 \\ 1 & 5 & -2 & 3 & -1 \\ \hline 0 & -4 & -2 & 7 & -1 \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \qquad B = \left[\begin{array}{cc} 6 & 4 \\ -2 & 1 \\ -3 & 7 \\ \hline -1 & 3 \\ 5 & 2 \end{array}\right] = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

$$AB = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} A_{11}B_1 + A_{12}B_2 \\ A_{21}B_1 + A_{22}B_2 \end{bmatrix}$$

only for square matrices

Invertibility of a matrix

An $n \times n$ matrix $\mathbf{A}$ is *invertible* if there is an $n \times n$ matrix $\mathbf{C}$ such that

$$\mathbf{CA} = \mathbf{I}_n \quad \text{and} \quad \mathbf{AC} = \mathbf{I}_n.$$

In this case $\mathbf{C}$ is the unique *inverse* of $\mathbf{A}$, which we denote by $\mathbf{A}^{-1}$.

Theorem (The left inverse is the right inverse)

If $\mathbf{A}$ *is* $n \times n$ *and there exists a matrix* $\mathbf{D}$ *such that* $\mathbf{DA} = \mathbf{I}_n$, *then* $\mathbf{AD} = \mathbf{I}_n$.

A matrix which is not invertible is called a *singular matrix*.

An invertible matrix is called a *nonsingular matrix*.

Theorem (Some properties of the inverse)

Let **A** *and* **B** *be invertible* $n \times n$ *matrices. Then*

1. $\mathbf{A}^{-1}$ *is invertible with* $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$.
2. $\mathbf{AB}$ *is invertible with* $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.
3. $\mathbf{A}^{T}$ *is invertible and* $(\mathbf{A}^{T})^{-1} = (\mathbf{A}^{-1})^{T}$.

Prove the above results.

① If A has inverse A^{-1}, then $A^{-1}A = I_n$ and $AA^{-1} = I_n$.

$\Rightarrow A$ is the inverse of A^{-1}.

$\Rightarrow (A^{-1})^{-1} = A$.

(2) $B^{-1}\underbrace{\underbrace{A^{-1}A}_{I_n}B}_{I_n} = I_n$, $\underbrace{A\underbrace{B B^{-1}}_{I_n}A^{-1}}_{I_n} = I_n$

So AB has inverse $B^{-1}A^{-1}$. $\Rightarrow (AB)^{-1} = B^{-1}A^{-1}$.

(3) We have:

$A^{-1}A = I_n$ and $AA^{-1} = I_n$

Now write

$$(A^{-1}A)^T = A^T(A^{-1})^T = I_n \qquad (I_n^T = I_n)$$

Also

$$(AA^{-1})^T = (A^{-1})^T A^T = I_n$$

So A^T has inverse $(A^{-1})^T$.

Theorem (Inverse of a 2×2 matrix)

Let $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. *If* $ad - bc \neq 0$, *then* $\mathbf{A}$ *is invertible and*

$$\mathbf{A}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $ad - bc = 0$ *then* $\mathbf{A}$ *is not invertible.*

Exercise: Find the inverse (if it exists) of each of the matrices

1. $\begin{bmatrix} 5 & 7 \\ -3 & -6 \end{bmatrix}^{-1} = \frac{1}{-30 - (-21)} \begin{bmatrix} -6 & -7 \\ 3 & 5 \end{bmatrix} = -\frac{1}{9} \begin{bmatrix} -6 & -7 \\ 3 & 5 \end{bmatrix}$
2. $\begin{bmatrix} -4 & 6 \\ 6 & -9 \end{bmatrix}$ ← "singular"

check $-\frac{1}{9}\begin{bmatrix} -6 & -7 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 5 & 7 \\ -3 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Lay, D. C. (2003). *Linear algebra and its applications. Third edition.* Pearson Education.