STAT 714 fa 2025

Linear algebra review 2/6

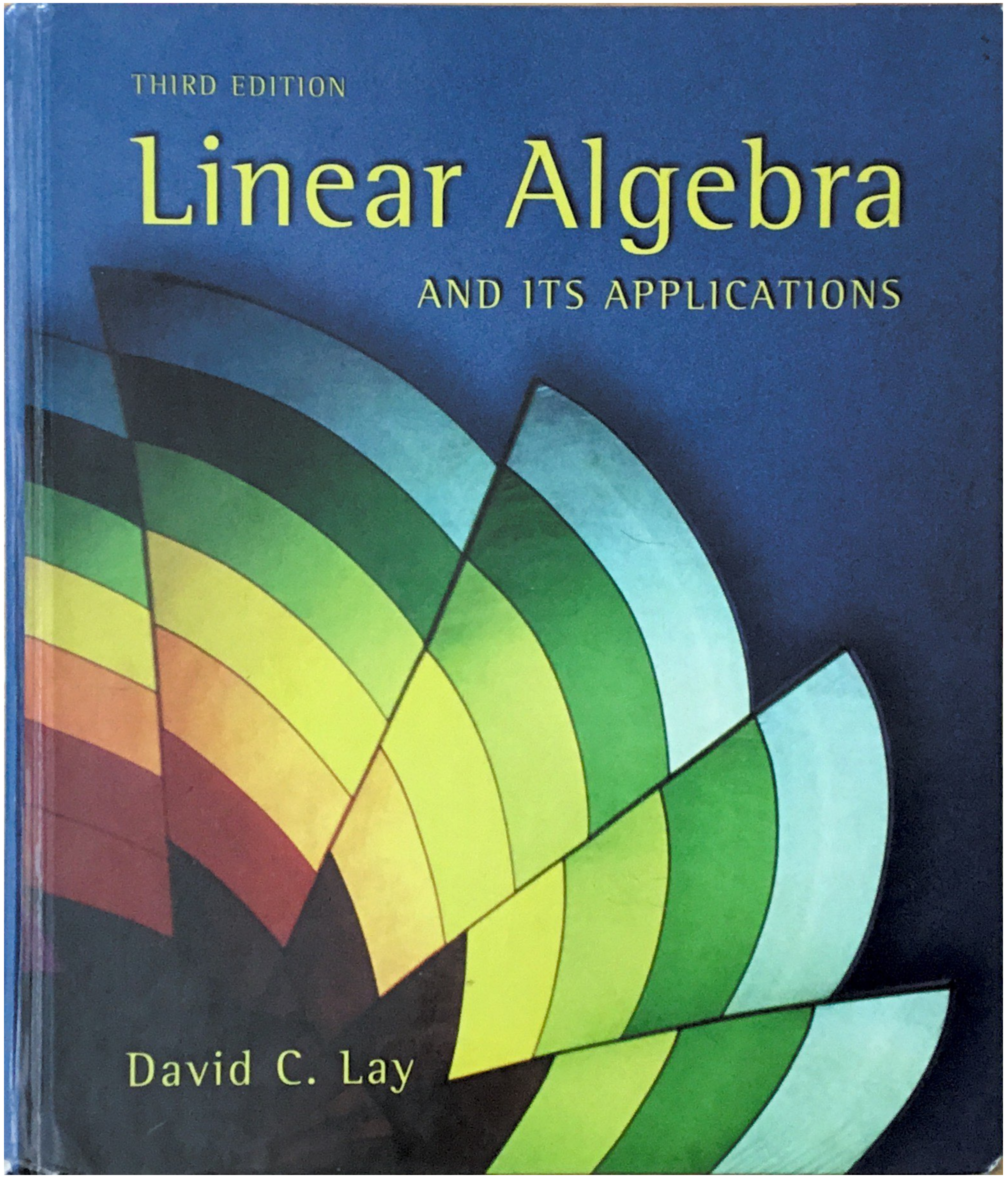
The equation $\mathbf{Ax} = \mathbf{b}$

Karl B. Gregory

University of South Carolina

These slides are an instructional aid; their sole purpose is to display, during the lecture, definitions, plots, results, etc. which take too much time to write by hand on the blackboard. They are not intended to explain or expound on any material.

These notes include scanned excerpts from Lay (2003):



1. The equation $\mathbf{Ax} = \mathbf{b}$
2. Elementary row operations and reduced row echelon form
3. Linear independence
4. Finding a matrix inverse with elementary row operations

Example problem: Give solution or characterize solutions if solvable...

$$
\begin{aligned}
x_1 + 3x_2 + x_3 &= 1 \\
-4x_1 - 9x_2 + 2x_3 &= -1 \\
-3x_2 - 5x_3 &= -3
\end{aligned}
$$

$$A \underset{\sim}{x} = \underset{\sim}{b}$$

$$\begin{bmatrix} 1 & 3 & 1 \\ -4 & -9 & 2 \\ 0 & -3 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$$

The equation $\mathbf{Ax} = \mathbf{b}$

We are often concerned with characterizing the solutions to $\mathbf{Ax} = \mathbf{b}$. It has either

- no solution,
- exactly one solution, or
- infinitely many solutions.

The equation $\mathbf{Ax} = \mathbf{b}$ is called *consistent* if at least one solution exists.

Homogeneous equation

A set of linear equations is called *homogeneous* if it can be written as $\boxed{\mathbf{Ax} = \mathbf{0}}$.

To which:

- The solution $\mathbf{x} = \mathbf{0}$ is called the *trivial solution*.
- A nonzero solution is called a *nontrivial solution*.

Example problem: Characterize solution(s) to $\mathbf{Ax} = \mathbf{b}$ if it is consistent, where

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 1 \\ -4 & -9 & 2 \\ 0 & -3 & -5 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}.$$

How? Use EROs to put $\boxed{\textit{augmented matrix } [\mathbf{A} \;\; \mathbf{b}]}$ in RREF...

1. The equation $\mathbf{Ax} = \mathbf{b}$
2. Elementary row operations and reduced row echelon form
3. Linear independence
4. Finding a matrix inverse with elementary row operations

Elementary row operations

1. Add to one row the multiple of another.
2. Interchange two rows.
3. Multiply a row by a scalar.

These will not change the set of solutions to a linear system of equations.

Example: Consider performing EROs on the system

$$\boxed{\begin{aligned} x_1 + 3x_2 + x_3 &= 1 \\ -4x_1 - 9x_2 + 2x_3 &= -1 \\ -3x_2 - 5x_3 &= -3 \end{aligned}}$$

put $[A \ \ b]$ in RREF:

$$\begin{bmatrix} 1 & 0 & \boxed{\ \ ?\ \ } & \\ 0 & 1 & & \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[A \ \ b] = \left[\begin{array}{ccc|c} 1 & 3 & 1 & 1 \\ -4 & -9 & 2 & -1 \\ 0 & -3 & -5 & -3 \end{array}\right]$$

add 4x 1st row to 2nd row

$$\stackrel{(\text{ERO})}{\sim} \left[\begin{array}{ccc|c} 1 & 3 & 1 & 1 \\ 0 & 3 & 6 & 3 \\ 0 & -3 & -5 & -3 \end{array}\right]$$

add 2nd row to 3rd row

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & 1 & 1 \\ 0 & 3 & 6 & 3 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

subtract 6x 3rd row from 2nd row

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & 1 & 1 \\ 0 & 3 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

divide 2nd row by 3

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

subtract $3\times$ 2nd row from 1st row

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

RREF

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{pivot columns}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

$$x_1 = -2$$

$$x_2 = 1$$

$$x_2 = 0$$

$$\underset{\sim}{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}.$$

Use elementary row operations to put [**A** **b**] in reduced row echelon form...

Reduced row echelon form

A matrix is in *row echelon form* if:

1. All nonzero rows are above all rows of all zeros.
2. Each leading entry (first nonzero entry) of a row is in a column to the right of the leading entry in the row above it.
3. All entries in a column below a leading entry are zeros.

A matrix is in *reduced row echelon form* if in addition to the above:

4. The leading entry of each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

Pivot position/column of a matrix

- A *pivot position* is a location in **A** which corresponds to the location of a leading 1 in a row echelon form of **A**.
- A *pivot column* is a column of **A** containing a pivot position.

Exercise: Put in RREF via EROs the augmented matrix corresponding to

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 1 \\ -4x_1 - 9x_2 + 2x_3 &= -1 \\ -3x_2 - 5x_3 &= -3 \end{aligned}$$

What is the solution?

Theorem (RREF and existence of solution to $\mathbf{Ax} = \mathbf{b}$)

1. *Each matrix is row-equivalent to exactly one reduced row echelon matrix.*
2. *An equation $\mathbf{Ax} = \mathbf{b}$ is consistent iff an echelon form of $[\mathbf{A}\ \mathbf{b}]$ has no row like*

$$[0 \ \cdots \ 0 \ b] \text{ with } b \text{ nonzero.}$$

Exercise: Check whether the system of equations is consistent.

$$\begin{aligned} 2x_1 + 2x_2 - 3x_3 &= 1 \\ -2x_2 + x_3 &= 0 \\ 4x_2 - 2x_3 &= 2 \end{aligned}$$

$$[\underset{\sim}{A} \ \ \underset{\sim}{b}] = \left[\begin{array}{ccc|c} 2 & 2 & -3 & 1 \\ 0 & -2 & 1 & 0 \\ 0 & 4 & -2 & 2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 2 & 2 & -3 & 1 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

Add 2x row 2 to row 3

Tells me the system is not consistent

$$\Rightarrow \quad \begin{aligned} 2x_1 + 2x_2 - 3x_3 &= 1 \\ -2x_2 + x_3 &= 0 \\ 0 &= 2 \text{ ?} \end{aligned}$$

impossible.

Inconsistent

Recipe for characterizing solutions when $\mathbf{Ax} = \mathbf{b}$ is consistent:

WRITING A SOLUTION SET (OF A CONSISTENT SYSTEM) IN PARAMETRIC VECTOR FORM

1. Row reduce the augmented matrix to reduced echelon form.
2. Express each basic variable in terms of any free variables appearing in an equation.
3. Write a typical solution $\mathbf{x}$ as a vector whose entries depend on the free variables, if any.
4. Decompose $\mathbf{x}$ into a linear combination of vectors (with numeric entries) using the free variables as parameters.

Note that non-pivot columns correspond to free variables.

Exercises: Give solution or characterize set of solutions if consistent:

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 3 \\ x_2 - x_3 &= 1 \\ x_1 \qquad + x_3 &= 1 \end{aligned}$$

$$[A \ \underset{\sim}{b}] = \left[\begin{array}{ccc|c} 2 & 1 & 1 & 3 \\ 0 & 1 & -1 & 1 \\ 1 & 0 & 1 & 1 \end{array}\right]$$

subtract from row 1

2x row 3

$$\sim \left[\begin{array}{ccc|c} 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 1 & 0 & 1 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 1 & 0 & 1 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

pivot columns   non-pivot column

$$\Rightarrow \quad \begin{aligned} x_1 \phantom{{}+x_2} + x_3 &= 1 \\ x_2 - x_3 &= 1 \end{aligned}$$

$$\Leftrightarrow \quad \begin{aligned} x_1 &= 1 - x_3 \\ x_2 &= 1 + x_3 \\ x_3 &= x_3 \quad (x_3 \text{ is ``free''}) \end{aligned}$$

The set of solutions is

$$\left\{ \underset{\sim}{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 - x_3 \\ 1 + x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \quad x_3 \in \mathbb{R} \right\}$$

$V = [\underset{\sim}{v}_1 \cdots \underset{\sim}{v}_p]$

$V\underset{\sim}{x} = x_1\underset{\sim}{v}_1 + \cdots + x_p\underset{\sim}{v}_p$

Linear independence of a set of vectors in $\mathbb{R}^n$

Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ be a set of vectors in $\mathbb{R}^n$. The set is

$V\underset{\sim}{x} = \underset{\sim}{0}$

- *linearly independent* if $\boxed{x_1\mathbf{v}_1 + \cdots + x_p\mathbf{v}_p = \mathbf{0}}$ has only the $\boxed{\text{trivial solution.}}$ all $x_1, \ldots, x_p$ are 0.
- *linearly dependent* if $x_1\mathbf{v}_1 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$ for some $x_1, \ldots, x_p$ not all zero.

(if $V\underset{\sim}{x} = \underset{\sim}{0}$ has a non-trivial solution)

Exercise: Check whether $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent, where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

$V\underset{\sim}{x} = \underset{\sim}{0}$

$$[V \;\; \underset{\sim}{0}] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 2 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\Rightarrow \quad \begin{aligned} x_1 &= 0 \\ x_2 &= 0 \\ x_3 &= 0 \end{aligned}$$

The only x_1, x_2, x_3 for which

$$\underset{\sim}{v}_1 x_1 + \underset{\sim}{v}_2 x_2 + \underset{\sim}{v}_3 x_3 = 0$$

is $x_1 = 0, \quad x_2 = 0, \quad x_3 = 0.$

$\Rightarrow \quad \{\underset{\sim}{v}_1, \underset{\sim}{v}_2, \underset{\sim}{v}_3\}$ is lin. indep.

Theorem (Characterization of linearly dependent sets)

If $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a linearly dependent set of nonzero vectors. Then at least one vector is a linear combination of the others.

Prove the result. Suppose $\{\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_p\}$ lin. dep.

Then $\underset{\sim}{v}_1 x_1 + \cdots + \underset{\sim}{v}_p x_p = 0$ for some $x_1, \ldots, x_p$ not all zero.

Suppose $x_1 \neq 0$. Then we can write

$$\underset{\sim}{v}_1 x_1 = -x_2 \underset{\sim}{v}_2 - \cdots - x_p \underset{\sim}{v}_p$$

$$\Rightarrow \quad \underset{\sim}{v}_1 = \left(-\frac{x_2}{x_1}\right) \underset{\sim}{v}_2 + \cdots + \left(-\frac{x_p}{x_1}\right) \underset{\sim}{v}_p$$

$$\underset{n \times p}{X}$$

$p = \#$ variables

$n = \#$ obs

$p > n$

Theorem (The $p > n$ theorem)

Any set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\boxed{\mathbb{R}^n}$ is linearly dependent if $p > n$.

Prove the result.

$$V\underset{\sim}{x} = \underset{\sim}{0} \quad \rightarrow \quad \begin{bmatrix} \underset{n \times p}{V} & \underset{\sim}{0} \end{bmatrix} \sim$$

$$\begin{array}{c} \\ n \text{ rows} \end{array} \left\{ \left[\begin{array}{cc|c} \overbrace{\text{Can have at most } n \text{ pivot columns}}^{n} & \overbrace{\text{non-pivots correspond to free variables}}^{p-n} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \end{array} \right] \right.$$

p columns

1. The equation $\mathbf{Ax} = \mathbf{b}$
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Proof: We have $A(A^{-1}\underset{\sim}{b}) = AA^{-1}\underset{\sim}{b} = I\underset{\sim}{b} = \underset{\sim}{b}$, so $\underset{\sim}{x} = A^{-1}\underset{\sim}{b}$ is a solution.

Suppose $\underset{\sim}{u}$ satisfies $A\underset{\sim}{u} = \underset{\sim}{b}$. Then $A^{-1}A\underset{\sim}{u} = A^{-1}\underset{\sim}{b}$

which gives $u = A^{-1}\underset{\sim}{b}$.

Theorem (Invertibility of $\mathbf{A}$ and the solution to $\mathbf{A}\mathbf{x} = \mathbf{b}$)

If $\mathbf{A}$ is an invertible $\boxed{n \times n}$ matrix, then for each $\mathbf{b} \in \mathbb{R}^n$ the equation $\mathbf{A}\mathbf{x} = \mathbf{b}$ has a unique solution $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$.

Prove the result.

$$\begin{bmatrix} 3 & 2 \\ -2 & 1 \end{bmatrix}^{-1} = \frac{1}{3-(-2)2}\begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}$$

Exercise: Find the solution to $\begin{bmatrix} 3 & 2 \\ -2 & 1 \end{bmatrix}\mathbf{x} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$ using the above result.

$$\underset{\sim}{x} = \frac{1}{7}\begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 5 \\ -1 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 7 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Theorem (Finding the inverse using EROs)

An $n \times n$ matrix $\mathbf{A}$ is invertible iff $\mathbf{A}$ is row equivalent to $\mathbf{I}_n$. In this case any sequence of EROs that reduces $\mathbf{A}$ to $\mathbf{I}_n$ transforms $\mathbf{I}_n$ into $\mathbf{A}^{-1}$.

Each ERO is equivalent to pre-multiplication by an *elementary matrix*.

Since EROs can be undone, elementary matrices are invertible.

Prove the above result.

If $E_k \cdots E_1 A = I$ then A has inverse given by $E_k \cdots E_1$

$$E_k \cdot \ldots \cdot E_1 I = A^{-1}$$

Exercise:

1. Give the matrix $\mathbf{E}$ such that $\mathbf{EA}$ performs on $\mathbf{A}$ the ERO "add to the second row three times the first row" when $\mathbf{A}$ is a 2×2 matrix.
2. Give the inverse of $\mathbf{E}$.

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}}_{E} \underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}_{A} = \begin{bmatrix} a_{11} & a_{12} \\ 3a_{11} + a_{21} & 3a_{12} + a_{22} \end{bmatrix}$$

$$E^{-1} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}^{-1} = \frac{1}{1\cdot 1 - 3\cdot 0} \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$$

$$E^{-1}(EA) = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ 3a_{11} + a_{21} & 3a_{12} + a_{22} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

ALGORITHM FOR FINDING A^{-1}

Row reduce the augmented matrix $[\,A \quad I\,]$. If A is row equivalent to I, then $[\,A \quad I\,]$ is row equivalent to $[\,I \quad A^{-1}\,]$. Otherwise, A does not have an inverse.

Exercise: Find (provided it exists) the inverse of the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}.$$

$$[A \;\; I] = \left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array}\right]$$

$R_3 = R_3 - 4R_2$

$$\sim \left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & -3 & -4 & 0 & -4 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{array}\right] \quad R_3 = R_3 + 3R_1$$

$$\sim \left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{array}\right] \quad R_2 = R_2 - 3R_3$$

$$\sim \left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & -9/2 & 7 & -3/2 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & -2 & 4 & -1 \\ 1 & 0 & 0 & -9/2 & 7 & -3/2 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -9/2 & 7 & -3/2 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{array}\right] \leftarrow A^{-1}$$

The Invertible Matrix Theorem

Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

(invertible = nonsingular)

a. A is an invertible matrix.

b. A is row equivalent to the $n \times n$ identity matrix.

c. A has n pivot positions.

d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

e. The columns of A form a linearly independent set.

f. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.

g. The equation $A\mathbf{x} = \mathbf{b}$ has ~~at least one~~ a unique solution for each $\mathbf{b}$ in $\mathbb{R}^n$.

h. The columns of A span $\mathbb{R}^n$.

i. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

j. There is an $n \times n$ matrix C such that $CA = I$.

k. There is an $n \times n$ matrix D such that $AD = I$.

l. A^T is an invertible matrix.

$$\underset{n\times n}{A} \sim \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$$

invertible

$$A\underset{\sim}{x} = \underset{\sim}{0}$$

$$\underset{\sim}{x} = A^{-1}\underset{\sim}{0} = \underset{\sim}{0}.$$

Lay, D. C. (2003). *Linear algebra and its applications. Third edition.* Pearson Education.