STAT 714 fa 2025

Linear algebra review 3/6

Column space, null space, and rank of a matrix

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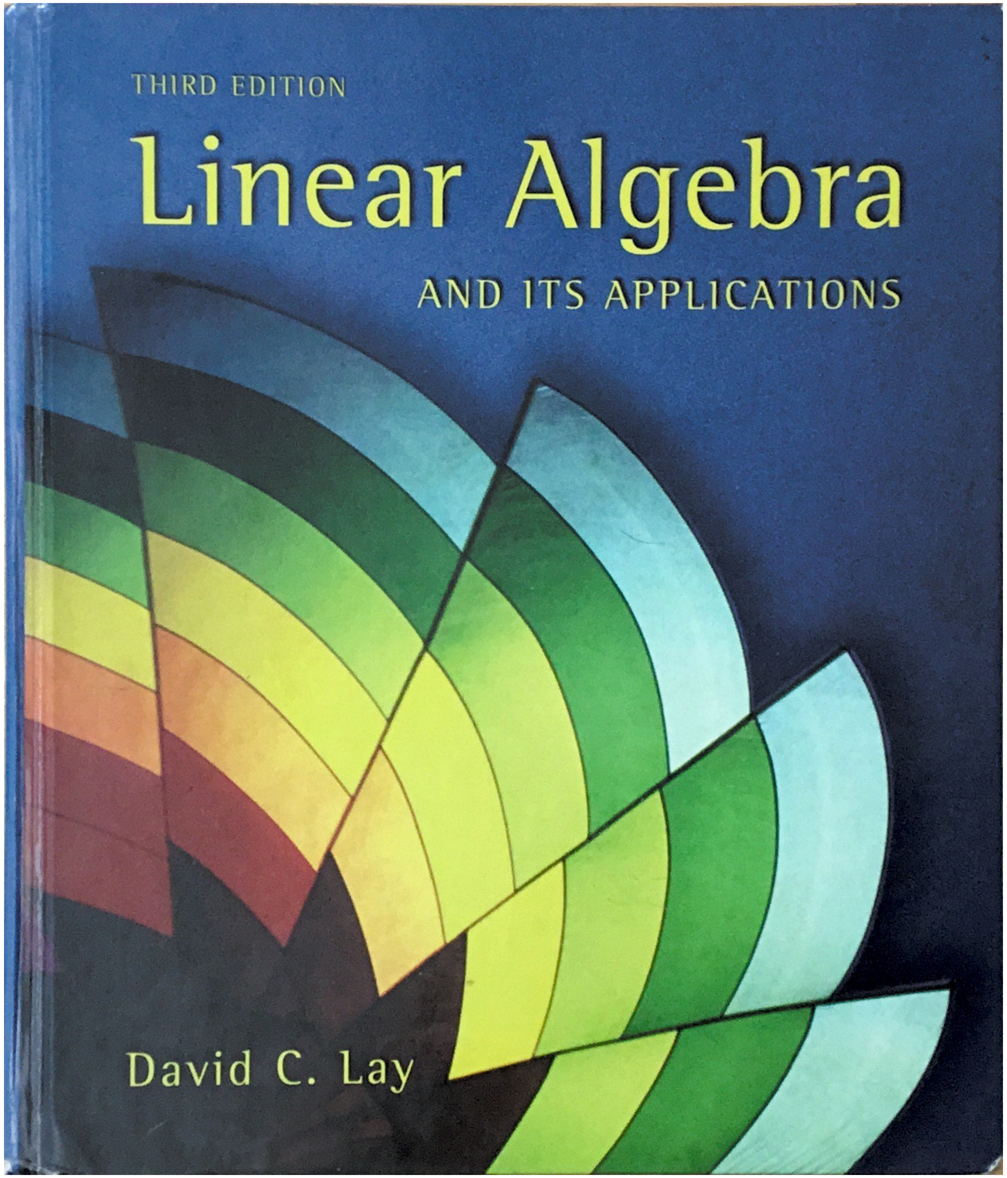
These slides are an instructional aid; their sole purpose is to display, during the lecture, definitions, plots, results, etc. which take too much time to write by hand on the blackboard. They are not intended to explain or expound on any material.

These notes include scanned excerpts from Lay (2003):



1. Vector spaces and subspaces
2. Null space and column space of a matrix
3. Bases and the dimension of a vector space
4. Rank of a matrix
5. Miscellaneous results

Vector space

A *vector space* is a nonempty set V of objects, called *vectors*, on which are defined two operations, called *addition* and *multiplication by scalars*, subject to these rules: For all $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and for all scalars c and d we must have

1. The sum of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u}+\mathbf{v}$, is in V.
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.
3. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$.
4. There is a **zero vector** $\mathbf{0}$ in V such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$.
5. For each $\mathbf{u}$ in V, there is a vector $-\mathbf{u}$ in V such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.
6. The scalar multiple of $\mathbf{u}$ by c, denoted by $c\mathbf{u}$, is in V.
7. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$.
8. $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$.
9. $c(d\mathbf{u})=(cd)\mathbf{u}$.
10. $1\mathbf{u}=\mathbf{u}$.

These imply the additional facts (i) $0\mathbf{u}=\mathbf{0}$, (ii) $c\mathbf{0}=\mathbf{0}$, and (iii) $-\mathbf{u}=(-1)\mathbf{u}$.

We will work in the vector space $\mathbb{R}^n$.

Subspace of a vector space

A *subspace* of a vector space V is a subset $H \subset V$ with three properties

1. The zero vector of V is in H.
2. For each $\mathbf{u}, \mathbf{v} \in H$, $\mathbf{u} + \mathbf{v} \in H$. *(Closure under vector addition)*
3. For each $\mathbf{u} \in H$ and $c \in \mathbb{R}$, $c\mathbf{u} \in H$. *(Closure under multiplication by scalars)*

$V = \mathbb{R}^2$

Exercise: For each subset of $\mathbb{R}^2$, determine if it is a subspace of $\mathbb{R}^2$:

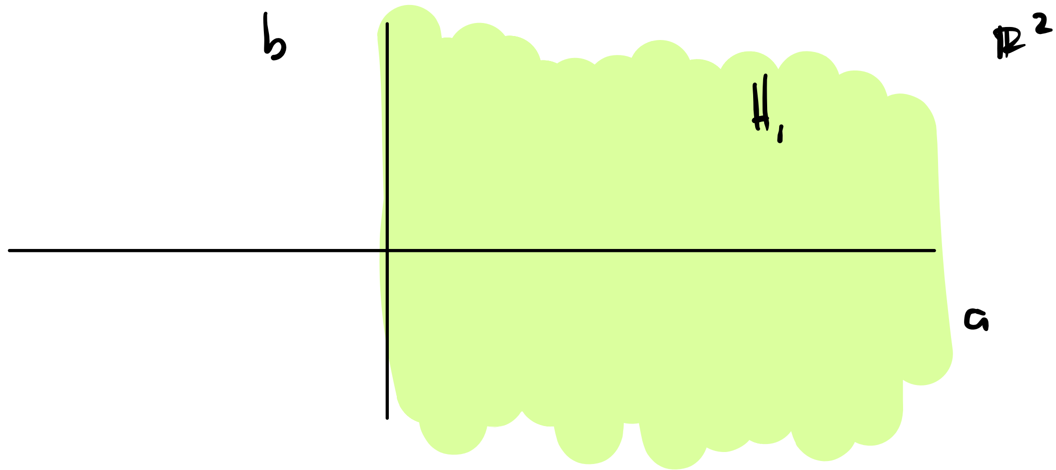
1. $$H_1 = \left\{ \left[\begin{array}{c} a \\ b \end{array} \right] : a \geq 0, b \in \mathbb{R} \right\}$$

(i) $\begin{bmatrix} 0 \\ 0 \end{bmatrix} \in H_1$ ✓ (iii) $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \in H_1$, but $(-1)\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \notin H_1$

2. $$H_2 = \left\{ \left[\begin{array}{c} x \\ y \end{array} \right] : y = 1 + x, \ x \in \mathbb{R} \right\}$$

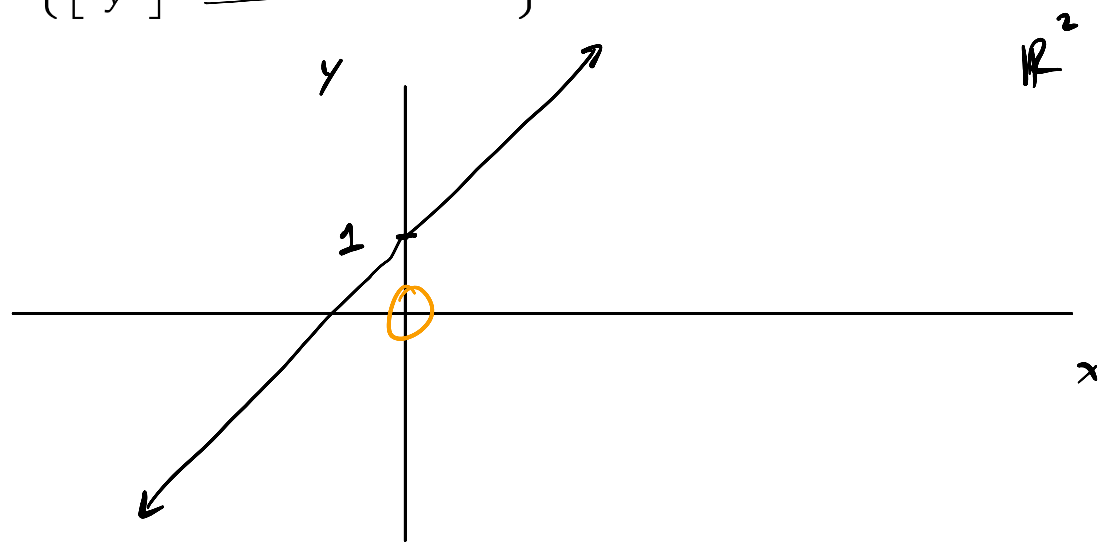
3. $$H_3 = \left\{ a \left[\begin{array}{c} 1 \\ 1 \end{array} \right] + b \left[\begin{array}{c} 1 \\ -1 \end{array} \right] : a, b \in \mathbb{R} \right\}$$

①



②

$$H_2 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : \underline{y = 1 + x}, \ x \in \mathbb{R} \right\}$$



$\underset{\sim}{0} \notin H_2$ so H_2 <u>not</u> a subspace

3 $H_3 = \left\{ a \begin{bmatrix} 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ -1 \end{bmatrix} : a, b \in \mathbb{R} \right\}$

(i) $\underset{\sim}{0} = 0 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \in H_3$ ✓

(ii) $\underset{\sim}{v}_1, \underset{\sim}{v}_2 \in H_3$

$$\underset{\sim}{v}_1 = a_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + b_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\underset{\sim}{v}_2 = a_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + b_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

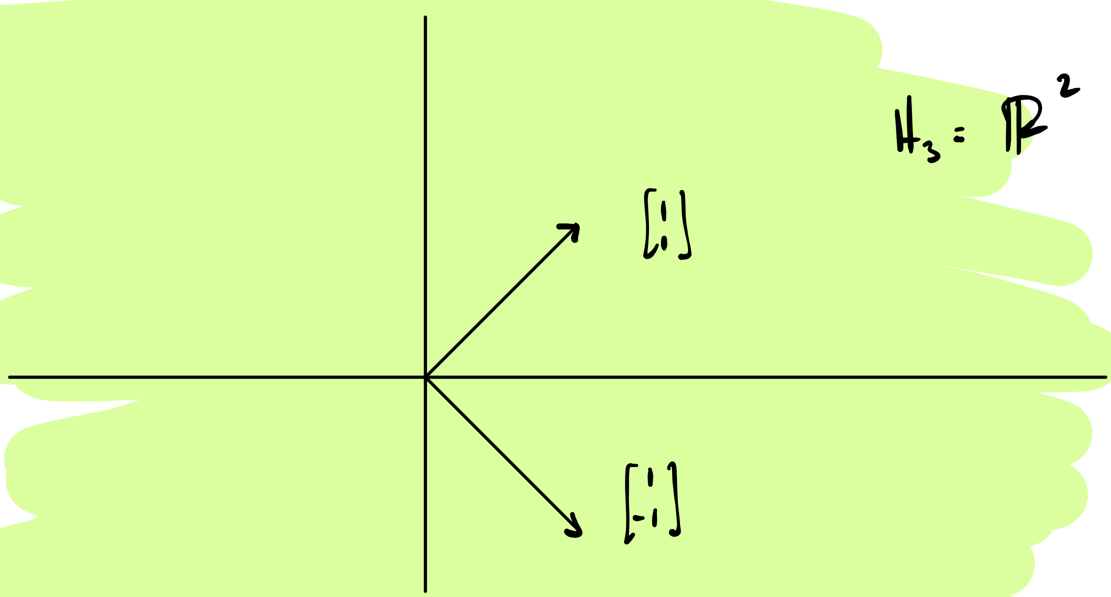
$$\underset{\sim}{v}_1 + \underset{\sim}{v}_2 = (a_1 + a_2) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (b_1 + b_2) \begin{bmatrix} 1 \\ -1 \end{bmatrix} \in H_3$$

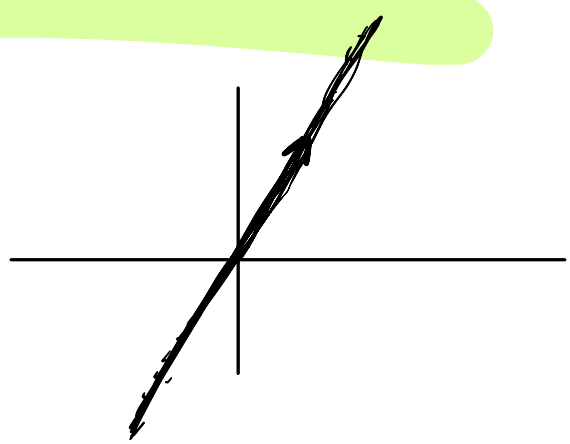
(iii) $\underset{\sim}{v} \in H_3$

$$\underset{\sim}{v} = a \begin{bmatrix} 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$c\underset{\sim}{v} = ca \begin{bmatrix} 1 \\ 1 \end{bmatrix} + cb \begin{bmatrix} 1 \\ -1 \end{bmatrix} \in H_3$$

$H_3 = \mathbb{R}^2$





A way to describe a subspace: the set of all linear combinations of a set of vectors.

Subspace of $\mathbb{R}^n$ spanned by a set of vectors

For $\mathbf{v}_1, \dots, \mathbf{v}_p \in \mathbb{R}^n$, denote the set of all linear combinations of $\mathbf{v}_1, \dots, \mathbf{v}_p$ by

$$\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\} = \{\mathbf{y} \in \mathbb{R}^n : \mathbf{y} = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p \text{ for some } c_1, \dots, c_p \in \mathbb{R}\}.$$

We call this set the *subspace of $\mathbb{R}^n$ spanned* by $\mathbf{v}_1, \dots, \mathbf{v}_p$.

Exercise: Depict $\text{Span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$. $= \left\{ \underset{\sim}{x} = a\begin{bmatrix} 1 \\ 1 \end{bmatrix} + b\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \ a,b \in \mathbb{R} \right\}$

Theorem (The span of a set of vectors makes a subspace)

If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are in a vector space V, then $\boxed{\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}}$ is a subspace of V.

$\underset{\sim}{v} \in \text{Span}\{\underset{\sim}{v}_1 \ldots, \underset{\sim}{v}_p\} \Rightarrow \underset{\sim}{v} = c_1\underset{\sim}{v}_1 + \cdots + c_p\underset{\sim}{v}_p$
for some $c_1, \ldots, c_p$.

Exercise: Prove the result.

(i) $\underset{\sim}{0} = 0\underset{\sim}{v}_1 + \cdots + 0\underset{\sim}{v}_p \in \text{Span}\{\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_p\}$.

(ii) $\underset{\sim}{a}, \underset{\sim}{b} \in \text{Span}\{\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_p\}$.

$\underset{\sim}{a} = c_1\underset{\sim}{v}_1 + \cdots + c_p\underset{\sim}{v}_p$

$\underset{\sim}{b} = d_1\underset{\sim}{v}_1 + \cdots + d_p\underset{\sim}{v}_p$

$\underset{\sim}{a} + \underset{\sim}{b} = (c_1 + d_1)\underset{\sim}{v}_1 + \cdots + (c_p + d_p)\underset{\sim}{v}_p \in \text{Span}\{\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_p\}$

(iii) $\underset{\sim}{a} \in \text{Span}\{\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_p\}$

$$c\,\underset{\sim}{a} = c c_1 \underset{\sim}{v}_1 + \cdots + c c_p \underset{\sim}{v}_p \in \text{Span}\{\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_p\}$$

Yes!

Exercise: Let $H = \{(a-3b, b-a, a, b)^T : a, b \in \mathbb{R}\}$. Check whether H is a subspace of $\mathbb{R}^4$. H[redacted]

$$H = \left\{ \begin{bmatrix} a-3b \\ b-a \\ a \\ b \end{bmatrix} : a, b \in \mathbb{R} \right\} = \left\{ a \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix} : a, b \in \mathbb{R} \right\}$$

$$= \text{Span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\underset{\sim}{v}_3 = \underset{\sim}{v}_2 - \underset{\sim}{v}_1$$

Exercise: For the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix},$$

check whether $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

" $\supset$ " $\quad \underset{\sim}{v} \in \text{Span}\{\underset{\sim}{v}_1, \underset{\sim}{v}_2\}$

$\Rightarrow \underset{\sim}{v} = c_1 \underset{\sim}{v}_1 + c_2 \underset{\sim}{v}_2 \qquad$ some $c_1, c_2 \in \mathbb{R}$

$\Rightarrow \underset{\sim}{v} = c_1 \underset{\sim}{v}_1 + c_2 \underset{\sim}{v}_2 + 0 \underset{\sim}{v}_3$

$\Rightarrow \underset{\sim}{v} \in \text{Span}\{\underset{\sim}{v}_1, \underset{\sim}{v}_2, \underset{\sim}{v}_3\}$

"C" $\quad \underset{\sim}{v} \in \operatorname{Span}\{\underset{\sim}{v}_1, \underset{\sim}{v}_2, \underset{\sim}{v}_3\}$

$$\Rightarrow \quad \underset{\sim}{v} = c_1 \underset{\sim}{v}_1 + c_2 \underset{\sim}{v}_2 + c_3 \underset{\sim}{v}_3 \qquad \text{some } c_1, c_2, c_3 \in \mathbb{R}$$

$$\Rightarrow \quad \underset{\sim}{v} = c_1 \underset{\sim}{v}_1 + c_2 \underset{\sim}{v}_2 + c_3 (\underset{\sim}{v}_2 - \underset{\sim}{v}_1)$$

$$\Rightarrow \quad \underset{\sim}{v} = (c_1 - c_3) \underset{\sim}{v}_1 + (c_2 + c_3) \underset{\sim}{v}_2$$

$$\Rightarrow \quad \underset{\sim}{v} \in \operatorname{Span}\{\underset{\sim}{v}_1, \underset{\sim}{v}_2\}.$$

$$\Rightarrow \quad \operatorname{Span}\{\underset{\sim}{v}_1, \underset{\sim}{v}_2, \underset{\sim}{v}_3\} = \operatorname{Span}\{\underset{\sim}{v}_1, \underset{\sim}{v}_2\}.$$

$\text{Nul } \underset{m\times n}{A} = \{ \underset{\sim}{x} : A\underset{\sim}{x} = \underset{\sim}{0} \} \subset \mathbb{R}^n$

Null space and column space of a matrix

Let $\mathbf{A}$ be an $m \times n$ matrix. Then

1. The *null space* $\text{Nul}\,\mathbf{A}$ of $\mathbf{A}$ is the set of all solutions to $\mathbf{Ax} = \mathbf{0}$.
2. The *column space* $\text{Col}\,\mathbf{A}$ of $\mathbf{A} = [\mathbf{a}_1, \ldots, \mathbf{a}_n]$ is $\text{Span}\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$.
3. The *row space* $\text{Row}\,\mathbf{A}$ of $\mathbf{A} = [\mathbf{r}_1, \ldots, \mathbf{r}_m]^T$ is $\text{Span}\{\mathbf{r}_1, \ldots, \mathbf{r}_m\}$.

$\underset{\sim}{a}_1 x_1 + \cdots + \underset{\sim}{a}_n x_n$

We can also write $\text{Col}\,\mathbf{A} = \{\mathbf{y} \in \mathbb{R}^m : \mathbf{y} = \mathbf{Ax} \text{ for some } \mathbf{x} \in \mathbb{R}^n\}$. $\subset \mathbb{R}^m$

Note that $\text{Row}\,\mathbf{A} = \text{Col}\,\mathbf{A}^T$.

$\text{Row } A^T = C$

Exercises:

1. Show that the null space of an $m \times n$ matrix $\mathbf{A}$ is a subspace of $\mathbb{R}^n$.
2. Show that the column space of an $m \times n$ matrix $\mathbf{A}$ is a subspace of $\mathbb{R}^m$.

① $\mathrm{Nul}\, A = \{\underset{\sim}{x} : A\underset{\sim}{x} = \underset{\sim}{0}\}$

(i) $A\underset{\sim}{0} = \underset{\sim}{0} \Rightarrow \underset{\sim}{0} \in \mathrm{Nul}\, A$

(ii) $\underset{\sim}{x}_1, \underset{\sim}{x}_2 \in \mathrm{Nul}\, A$

$\Rightarrow A\underset{\sim}{x}_1 = \underset{\sim}{0} \quad , \quad A\underset{\sim}{x}_2 = \underset{\sim}{0}$

$\Rightarrow A(\underset{\sim}{x}_1 + \underset{\sim}{x}_2) = A\underset{\sim}{x}_1 + A\underset{\sim}{x}_2 = \underset{\sim}{0}$

$\Rightarrow \underset{\sim}{x}_1 + \underset{\sim}{x}_2 \in \mathrm{Nul}\, A.$

(iii) $\underset{\sim}{x} \in \mathrm{Nul}\, A$

$\Rightarrow A\underset{\sim}{x} = \underset{\sim}{0}$

$\Rightarrow A(c\underset{\sim}{x}) = cA\underset{\sim}{x} = \underset{\sim}{0} \qquad c \in \mathbb{R}.$

$\Rightarrow c\underset{\sim}{x} \in \mathrm{Nul}\, A.$

$$\text{Col } A = \text{Span}\{\underset{\sim}{v}_1, \underset{\sim}{v}_2, \underset{\sim}{v}_3\} = \text{Span}\{\underset{\sim}{v}_1, \underset{\sim}{v}_2\} = \text{Span}\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}\right\}$$

Exercise: Give the null space and column space of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix} = [\underset{\sim}{v}_1 \; \underset{\sim}{v}_2 \; \underset{\sim}{v}_3]$$

Write each as the span of a set of vectors.

$$\text{Nul } A = \{ \underset{\sim}{x} : A\underset{\sim}{x} = \underset{\sim}{0} \}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 2 & 0 \end{array}\right] \underset{R_3 - R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \quad R_2 - R_3$$

$$\sim \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \quad R_1 - R_2$$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \quad R_2 - R_3$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \text{rearrange}$$

$$\underbrace{}_{\text{pivot}} \qquad \underbrace{}_{\text{non-pivot}}$$

$$\rightarrow \quad \begin{aligned} x_1 \quad - x_3 &= 0 \\ x_2 + x_3 &= 0 \\ x_3 &= \text{"free"} \end{aligned}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \quad \text{Nul } A = \text{Span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix} c \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \underset{\sim}{0} \quad \text{for all } c \in \mathbb{R}.$$

Contrast Between Nul A and Col A for an $m \times n$ Matrix A

Nul A	Col A
1. Nul A is a subspace of $\mathbb{R}^n$.	**1.** Col A is a subspace of $\mathbb{R}^m$.
2. Nul A is implicitly defined; that is, you are given only a condition ($A\mathbf{x} = \mathbf{0}$) that vectors in Nul A must satisfy.	**2.** Col A is explicitly defined; that is, you are told how to build vectors in Col A.
3. It takes time to find vectors in Nul A. Row operations on $[\,A \quad \mathbf{0}\,]$ are required.	**3.** It is easy to find vectors in Col A. The columns of A are displayed; others are formed from them.
4. There is no obvious relation between Nul A and the entries in A.	**4.** There is an obvious relation between Col A and the entries in A, since each column of A is in Col A.
5. A typical vector $\mathbf{v}$ in Nul A has the property that $A\mathbf{v} = \mathbf{0}$.	**5.** A typical vector $\mathbf{v}$ in Col A has the property that the equation $A\mathbf{x} = \mathbf{v}$ is consistent.
6. Given a specific vector $\mathbf{v}$, it is easy to tell if $\mathbf{v}$ is in Nul A. Just compute $A\mathbf{v}$.	**6.** Given a specific vector $\mathbf{v}$, it may take time to tell if $\mathbf{v}$ is in Col A. Row operations on $[\,A \quad \mathbf{v}\,]$ are required.
7. Nul $A = \{\mathbf{0}\}$ if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.	**7.** Col $A = \mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^m$.
8. Nul $A = \{\mathbf{0}\}$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.	**8.** Col $A = \mathbb{R}^m$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ *onto* $\mathbb{R}^m$.

Basis for a vector space

Let H be a subsp. of a vec. sp. V and $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ a set of vectors in V. If

1. $\mathcal{B}$ is a linearly independent set, and
2. $H = \text{Span}\{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$,

then $\mathcal{B}$ is called a *basis* for H.

Example: The columns of the $n \times n$ identity matrix, that is the set of vectors

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots \quad, \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

is called the *standard basis* for $\mathbb{R}^n$.

Exercise: Check the following:

No, lin. dep.

1. Is the set of vectors $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$ a basis for $\mathbb{R}^3$?

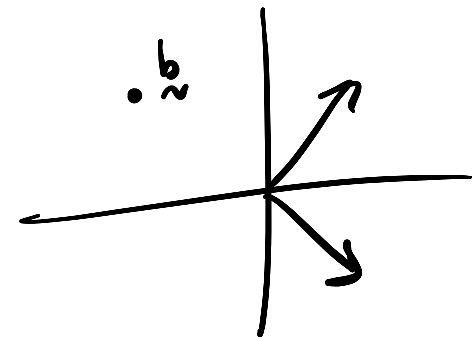
2. Do the columns of the matrix $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ form a basis for $\mathbb{R}^2$? ✓

(i) Lin indep. ✓

(ii) $\text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} = \mathbb{R}^2$

"$\subset$" ✓

"$\supset$" Take $\underset{\sim}{b} \in \mathbb{R}^2$



Can I find x_1 and x_2 such that

$$\underset{\sim}{b} = x_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad ?$$

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \underset{\sim}{x} = \underset{\sim}{b} \qquad \underset{\sim}{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\underset{\sim}{x} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \underset{\sim}{b}$$

$$= \frac{1}{1(-1)-1} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \underset{\sim}{b}$$

$$= -\frac{1}{2} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \underset{\sim}{b}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \underset{\sim}{b}$$

$$= \begin{bmatrix} \frac{1}{2}(b_1 + b_2) \\ \frac{1}{2}(b_1 - b_2) \end{bmatrix}$$

$$\Rightarrow \quad \underset{\sim}{b} = \frac{1}{2}(b_1 + b_2) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{2}(b_1 - b_2) \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\Rightarrow \quad \mathbb{R}^2 \subset \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}.$$

Spanning sets can have "extra", unneeded vectors in them:

Theorem (Spanning set theorem)

Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ be a set of vectors in V and let $H = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

1. *If any vector in $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a linear combination of the others, it can be removed, and the resulting set of vectors will still span H.*
2. *If $H \neq \{\mathbf{0}\}$, then some subset of $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a basis for H.*

Prove the result.

1 Suppose $\underset{\sim}{v}_1 = c_2 \underset{\sim}{v}_2 + \cdots + c_p \underset{\sim}{v}_p$ [W.L.O.G. assume it's $\underset{\sim}{v}_1$]

$\text{Span}\{\underset{\sim}{v}_2, \ldots, \underset{\sim}{v}_p\} = \text{Span}\{\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_p\}$

"$\subseteq$" is simple

"$\supseteq$" Take $\underset{\sim}{v} \in \operatorname{Span}\{\underset{\sim}{v}_1, \dots, \underset{\sim}{v}_p\}$

$$\Rightarrow \quad \underset{\sim}{v} = d_1 \underset{\sim}{v}_1 + d_2 \underset{\sim}{v}_2 + \dots + d_p \underset{\sim}{v}_p$$

$$\Rightarrow \quad \underset{\sim}{v} = d_1 \left(c_2 \underset{\sim}{v}_2 + \dots + c_p \underset{\sim}{v}_p\right) + d_2 \underset{\sim}{v}_2 + \dots + d_p \underset{\sim}{v}_p$$

$$\Rightarrow \quad \underset{\sim}{v} = (d_1 c_2 + d_2) \underset{\sim}{v}_2 + \dots + (d_1 c_p + d_p) \underset{\sim}{v}_p$$

$$\Rightarrow \quad \underset{\sim}{v} \in \operatorname{Span}\{\underset{\sim}{v}_2, \dots, \underset{\sim}{v}_p\}.$$

Theorem (Find a basis for the column space of a matrix)

1. *If a matrix* $\mathbf{A}$ *can be transformed to* $\mathbf{B}$ *with EROs then* $\operatorname{Nul}\mathbf{A} = \operatorname{Nul}\mathbf{B}$.
2. *The pivot columns of a matrix* $\mathbf{A}$ *form a basis for* $\operatorname{Col}\mathbf{A}$.

Discuss the result.

Col A

Exercise: Construct a basis for [redacted] where

[redacted]

non-pivot columns

$$A = \begin{bmatrix} 1 & 6 & 2 & -4 \\ -3 & 2 & -2 & -8 \\ 4 & -1 & 3 & 9 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 6 & 2 & -4 \\ 0 & 20 & 4 & -20 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot columns   pivot colums

$$\Rightarrow \quad \text{Col } A = \text{Span}\left\{ \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}, \begin{bmatrix} 6 \\ 2 \\ -1 \end{bmatrix} \right\}$$

Theorem (Find a basis for the column space of a matrix)

1. *If a matrix* $\mathbf{A}$ *can be transformed to* $\mathbf{B}$ *with EROs then* $\text{Nul}\,\mathbf{A} = \text{Nul}\,\mathbf{B}$. ✓
2. *The pivot columns of a matrix* $\mathbf{A}$ *form a basis for* $\text{Col}\,\mathbf{A}$.

1 proof

$$\underbrace{E_k \cdots E_1}_{E} A = B,$$

We have $EA = B$, and $A = E^{-1}B$.

"$\subset$" $\quad \underset{\sim}{x} \in \text{Nul } A \Rightarrow A\underset{\sim}{x} = \underset{\sim}{0}$

$\Rightarrow EA\underset{\sim}{x} = E\underset{\sim}{0}$

$\Rightarrow B\underset{\sim}{x} = \underset{\sim}{0}$

$\Rightarrow \underset{\sim}{x} \in \text{Nul } B.$

"$\supset$" $\quad \underset{\sim}{x} \in \text{Nul } B \Rightarrow B\underset{\sim}{x} = \underset{\sim}{0}$

$\Rightarrow E^{-1}B\underset{\sim}{x} = E^{-1}\underset{\sim}{0}$

$\Rightarrow A\underset{\sim}{x} = \underset{\sim}{0}$

$\Rightarrow \underset{\sim}{x} \in \text{Nul } A.$

2. *The pivot columns of a matrix* **A** *form a basis for* Col **A**.

Let's say $A \overset{\text{EROs}}{\sim} B$, B is in an echelon form.

Suppose r columns of B are pivot columns, collect them in the matrix B_r.

Take same cols of A, put them in A_r.

$$A_r \overset{\text{EROs}}{\sim} B_r \, .$$

Now by (1) $\text{Nul } A_r = \text{Nul } B_r$,

Note: set of pivot columns in B_r is always lin. indep.

So $\text{Nul } B_r = \{ \underset{\sim}{0} \}$.

Therefore $\text{Nul } A_r = \{ \underset{\sim}{0} \}$,

so the columns of A_r are linearly <u>indep.</u>

Let $\underset{\sim}{a}$ be (a non-pivot) some column of A not in A_r.

Write

$$\begin{bmatrix} \underbrace{A_r}_{\text{pivot cols}} & \underset{\text{a non-pivot column}}{\underset{\uparrow}{\underset{\sim}{a}}} \end{bmatrix} \overset{\text{EROs}}{\sim} \begin{bmatrix} \underbrace{B_r}_{\text{pivot columns}} & \underset{\text{non-pivot column.}}{\underset{\uparrow}{\underset{\sim}{b}}} \end{bmatrix}$$

I have $\text{Nul } [B_r \ \underset{\sim}{b}]$ contains a nonzero vector, Therefore $\quad \mathrm{Nul}\,[A_r \ \underset{\sim}{a}] = \mathrm{Nul}\,[B_r \ \underset{\sim}{b}]$ also

contains a nonzero vector.

Tells us that the columns of $[A_r \ \underset{\sim}{a}]$ are <u>not</u>

linearly <u>indep</u>.

$\Rightarrow$ columns of A_r form a basis for

Col A.

Theorem (Unique representation theorem)

Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be a basis for a vector space V. Then for each $\mathbf{x}$ in V, there exists a unique set of scalars $c_1, \dots, c_n$ such that

$$\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n.$$

Coordinates with respect to a basis

For the above we may write $\mathbf{x} = [\mathbf{b}_1 \cdots \mathbf{b}_n][\mathbf{x}]_{\mathcal{B}}$, where $[\mathbf{x}]_{\mathcal{B}} = (c_1, \dots, c_n)^T$ is the *coordinate vector of* $\mathbf{x}$ *relative to the basis* $\mathcal{B}$.

Prove the unique representation theorem.

U.R.T. Let $\mathcal{B} = \{\underset{\sim}{b}_1, \dots, \underset{\sim}{b}_n\}$ be a basis for V.

linearly independent $\Rightarrow$ $B\underset{\sim}{x} = \underset{\sim}{0}$ has only the trivial solution $\underset{\sim}{x} = \underset{\sim}{0}$.

Set $B = [\underset{\sim}{b}_1 \cdots \underset{\sim}{b}_n]$. Then for $x \in V$, there exists a vector $\underset{\sim}{c}$ such that

$$\underset{\sim}{x} = B\underset{\sim}{c}.$$

Suppose $\underset{\sim}{c}_1$ and $\underset{\sim}{c}_2$ are vectors such that

$$\underset{\sim}{x} = B\underset{\sim}{c}_1 \quad \text{and} \quad \underset{\sim}{x} = B\underset{\sim}{c}_2.$$

Then
$$B\underset{\sim}{c}_1 = B\underset{\sim}{c}_2$$
$$\Leftrightarrow \quad B\underset{\sim}{c}_1 - B\underset{\sim}{c}_2 = \underset{\sim}{0}$$
$$\Leftrightarrow \quad B(\underset{\sim}{c}_1 - \underset{\sim}{c}_2) = \underset{\sim}{0}$$

Nul $B = \{0\}$, since B has linearly indep. cols.

$$\Leftrightarrow \quad \underset{\sim}{c}_1 - \underset{\sim}{c}_2 = \underset{\sim}{0}$$
$$\Leftrightarrow \quad \underset{\sim}{c}_1 = \underset{\sim}{c}_2.$$

The following results allow us to define the *dimension* of a vector space.

Theorem (Dimension theorem)

Let V be a vector space and let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be a basis for V.

1. *Any set of more than n vectors in V is linearly dependent.*
2. *Every basis for V consists of exactly n vectors.*

Prove the dimension theorem.

Dimension of a vector space

Let V be a vector space.

1. If V is spanned by a finite set, then V is *finite-dimensional*.
2. If V is not spanned by any finite set, then V is *infinite-dimensional*.
3. The *dimension* $\boxed{\dim V}$ of V is the number of vectors in a basis for V.
4. If $V = \{\mathbf{0}\}$ then we define $\dim V = 0$

Exercise: Give the dimension of the space $\text{Span}\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}0\\1\\2\end{bmatrix}\right\}$.

dim = 2

To summarize some of the foregoing results:

How do you know you have a basis?

For a p-dimensional vector space V:

1. Any set of p linearly independent vectors in V is a basis for V.
2. Any set of p vectors that spans V is a basis for V.

linearly indep.

correspond to non-pivot columns

Result (Relating dimensions to $\mathbf{Ax} = \mathbf{0}$ and the echelon form)

1. $\dim \mathrm{Nul}\, \mathbf{A}$ *is the number of free variables in* $\mathbf{Ax} = \mathbf{0}$.
2. $\dim \mathrm{Col}\, \mathbf{A}$ *is the number of pivot columns in* $\mathbf{A}$.

Implies that $\dim \mathrm{Col}\, \mathbf{A}$ and $\dim \mathrm{Nul}\, \mathbf{A}$ add up to the number of columns of $\mathbf{A}$.

Discuss results from an echelon form perspective.

Exercise: Give the dimension of the column space and the null space of the matrix

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

$$\dots \sim \begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis for Col A is $\left\{ \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} \right\}$.

$\Rightarrow \dim \mathrm{Col}\, A = 2$

$$\dim \operatorname{Nul} A = 3$$

$$\sim \begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & \underline{0} & \underline{0} \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 - 2x_2 \quad\quad - x_4 + 3x_5 &= 0 \\ x_3 + 2x_4 - 2x_5 &= 0 \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

$$\operatorname{Nul} A = \operatorname{Span} \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Row A = span {rows of A}

Result (Basis for row space of a matrix)

If $\mathbf{A}$ *and* $\mathbf{B}$ *are row-equivalent (can do EROs to transform* $\mathbf{A}$ *into* $\mathbf{B}$*) then*

1. $\boxed{\text{Row}\,\mathbf{A} = \text{Row}\,\mathbf{B}}$.
2. *The nonzero rows of* $\mathbf{B}$ *form a basis for* Row $\mathbf{A}$ *as well as for* Row $\mathbf{B}$.

dim Row A = # nonzero rows = # pivot columns = dim Col A

Discuss results.

Exercise: Find bases for the row space, column space, and null space of the matrix

$$\underbrace{\begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}}_{A} \sim \underbrace{\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{B}$$

lin. indep. rows. These form a basis for row A. and Row B.

$$\text{Row } A = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -5 \end{bmatrix} \right\}$$

$$\text{Col } A = \text{Span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 11 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 7 \\ 5 \end{bmatrix} \right\}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 \quad + x_3 \quad x_5 = 0$$

$$x_2 \quad -2x_3 \quad 3x_5 = 0$$

$$x_4 \quad -5x_5 = 0$$

$$\begin{aligned} x_1 &= -x_3 - x_5 \\ x_2 &= 2x_3 - 3x_5 \\ x_3 &= x_3 \\ x_4 &= 5x_5 \\ x_5 &= x_5 \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix}$$

$$\dim \text{Nul } A = 2$$

$$\text{So} \quad \text{Nul } A = \text{Span}\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix} \right\}.$$

Rank of a matrix

The *rank* of a matrix is the dimension of its col. space. Write rank $\mathbf{A}$ = dim Col $\mathbf{A}$.

① rank A^T = dim Col A^T = dim Row A = dim Col A = rank A.

Theorem (Rank Theorem)

Let $\mathbf{A}$ *be an* $m \times n$ *matrix. Then*

$A_{m\times n} = \overbrace{\left[\quad\quad\quad\right]}^{n}$

pivot-columns, non-pivot columns.

1. rank $\mathbf{A}$ = rank $\mathbf{A}^T$
2. $\boxed{\text{rank }\mathbf{A}} + \boxed{\text{dim Nul }\mathbf{A}} = n$

\# non-pivot columns

dim Col A = # pivot columns

A matrix has *full-column rank* if its rank is equal to its number of columns.

Discuss echelon-form arguments for the rank theorem.

The Invertible Matrix Theorem (continued)

Let A be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix.

m. The columns of A form a basis of $\mathbb{R}^n$.

n. $\operatorname{Col} A = \mathbb{R}^n$

o. $\dim \operatorname{Col} A = n$

p. $\operatorname{rank} A = n$

q. $\operatorname{Nul} A = \{\mathbf{0}\}$

r. $\dim \operatorname{Nul} A = 0$

Theorem (cf. Results A.1 and A.2 in Monahan (2008))

1. *We have* $\operatorname{Col}\mathbf{A} \subset \operatorname{Col}\mathbf{B}$ *if and only if* $\mathbf{A} = \mathbf{B}\mathbf{C}$ *for some matrix* $\mathbf{C}$.
2. $\operatorname{rank}(\mathbf{A}\mathbf{B}) \leq \min\{\operatorname{rank}\mathbf{A}, \operatorname{rank}\mathbf{B}\}$.
3. *If* $\mathbf{A}$ *has full-column rank, then* $\operatorname{Nul}\mathbf{A} = \{\mathbf{0}\}$.

Prove the above results.

Theorem (cf. Result A.8, Cor A.1, A.2, and Lemma A.1 of Monahan)

1. *If* $\mathbf{Ax} + \mathbf{b} = \mathbf{0}$ *for all* $\mathbf{x} \in \mathbb{R}^n$ *then* $\mathbf{A} = \mathbf{0}$ *and* $\mathbf{b} = \mathbf{0}$.
2. *If* $\mathbf{Bx} = \mathbf{Cx}$ *for all* $\mathbf{x} \in \mathbb{R}^n$ *then* $\mathbf{B} = \mathbf{C}$.
3. *If* $\mathbf{A}$ *has full-column rank and* $\mathbf{AB} = \mathbf{AC}$ *then* $\mathbf{B} = \mathbf{C}$.
4. *If* $\mathbf{C}^T\mathbf{C} = \mathbf{0}$ *then* $\mathbf{C} = \mathbf{0}$.

Prove the above results.

Lay, D. C. (2003). *Linear algebra and its applications. Third edition.* Pearson Education.

Monahan, J. F. (2008). *A primer on linear models*. CRC Press.