STAT 714 fa 2025

Linear algebra review 4/6

Orthogonal subspaces, bases, projections, Gram-Schmidt

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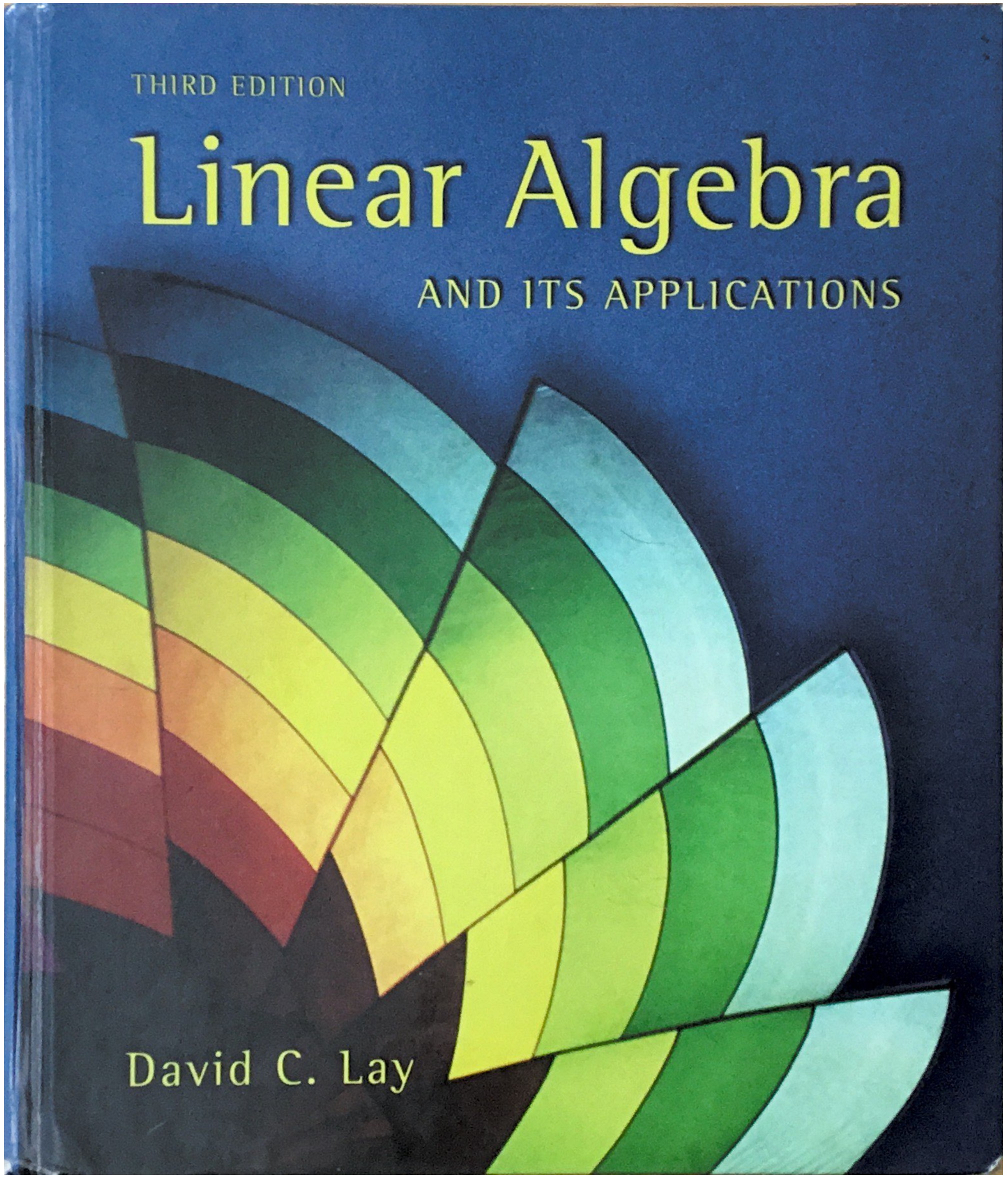
These slides are an instructional aid; their sole purpose is to display, during the lecture, definitions, plots, results, etc. which take too much time to write by hand on the blackboard. They are not intended to explain or expound on any material.

These notes include scanned excerpts from Lay (2003):



$\underset{\sim}{u} \cdot \underset{\sim}{v} = \underset{\sim}{u}^T \underset{\sim}{v} = 0$, orthogonal

Orthogonal and orthonormal sets of vectors

A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\} \in \mathbb{R}^n$ is an *orthogonal set* if $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ for all $i \neq j$.

If in addition $\|\mathbf{v}_i\| = 1$ for $i = 1, \ldots, n$, the set is an *orthonormal set*.

$\|\underset{\sim}{v}\| = \sqrt{\underset{\sim}{v} \cdot \underset{\sim}{v}} = \sqrt{\underset{\sim}{v}^T \underset{\sim}{v}}$

Example: The *elementary vectors*

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \quad \ldots \quad , \quad \mathbf{e}_{n-1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

in $\mathbb{R}^n$ are an orthonormal set; moreover $\text{Span}\{\mathbf{e}_1, \ldots, \mathbf{e}_n\} = \mathbb{R}^n$.

$$\underset{m\times n}{\mathbf{U}} = [\underset{\sim}{u}_1 \cdots \underset{\sim}{u}_n] \qquad \mathbf{U}^T\mathbf{U} = \begin{bmatrix} \underset{\sim}{u}_1^T \\ \vdots \\ \underset{\sim}{u}_n^T \end{bmatrix} [\underset{\sim}{u}_1 \cdots \underset{\sim}{u}_n] = \begin{bmatrix} \underset{\sim}{u}_1^T\underset{\sim}{u}_1 & \underset{\sim}{u}_1^T\underset{\sim}{u}_2 & \cdots & \underset{\sim}{u}_1^T\underset{\sim}{u}_n \\ \underset{\sim}{u}_2^T\underset{\sim}{u}_1 & \underset{\sim}{u}_2^T\underset{\sim}{u}_2 & \cdots & \underset{\sim}{u}_2^T\underset{\sim}{u}_n \\ \vdots & \vdots & \ddots & \vdots \\ \underset{\sim}{u}_n^T\underset{\sim}{u}_1 & \underset{\sim}{u}_n^T\underset{\sim}{u}_2 & \cdots & \underset{\sim}{u}_n^T\underset{\sim}{u}_n \end{bmatrix} = \mathbf{I}_n$$

Result (Orthonormal columns)

An $m \times n$ matrix $\mathbf{U}$ has $\boxed{\text{orthonormal}}$ columns if and only if $\mathbf{U}^T\mathbf{U} = \mathbf{I}$.

Prove the result.

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = u_1 v_1 + \cdots + u_n v_n$$

Exercise: Check if these matrices have orthonormal columns:

$$\boxed{\mathbf{C} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}} \text{ Yes} \qquad \boxed{\mathbf{D} = \begin{bmatrix} 1 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} \end{bmatrix}} \text{ No}$$

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}^T \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = 1 \qquad \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}^T \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} = 0$$

$$\underset{\sim}{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \underset{\sim}{u} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \qquad \underset{\sim}{u} \cdot \underset{\sim}{v} = 1 \cdot \frac{1}{\sqrt{2}} + 0 \, \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}.$$

Theorem (Results for matrices with orthonormal columns)

Let $\mathbf{U}$ *be an* $m \times n$ *matrix with orthonormal columns and let* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. *Then*

1. $\|\mathbf{U}\mathbf{x}\| = \|\mathbf{x}\|$
2. $(\mathbf{U}\mathbf{x}) \cdot (\mathbf{U}\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$
3. $\boxed{(\mathbf{U}\mathbf{x}) \cdot (\mathbf{U}\mathbf{y}) = 0 \iff \mathbf{x} \cdot \mathbf{y} = 0}$ follows from (2)

Exercise: Prove the above results.

(1) $\|U\underset{\sim}{x}\| = \sqrt{(U\underset{\sim}{x})^T U\underset{\sim}{x}} = \sqrt{\underset{\sim}{x}^T \underbrace{U^T U}_{I} \underset{\sim}{x}} = \sqrt{\underset{\sim}{x}^T \underset{\sim}{x}} = \|\underset{\sim}{x}\|$

(2) $(U\underset{\sim}{x}) \cdot (U\underset{\sim}{y}) = (U\underset{\sim}{x})^T (U\underset{\sim}{y}) = \underset{\sim}{x}^T \underbrace{U^T U}_{I} \underset{\sim}{y} = \underset{\sim}{x}^T \underset{\sim}{y} = \underset{\sim}{x} \cdot \underset{\sim}{y}.$

$$\underset{n\times n}{U} \qquad \boxed{\underset{n\times n}{U^T}\underset{n\times n}{U} = I} \qquad \boxed{\underset{n\times n}{U}\underset{n\times n}{U^T} = I_n}$$

Orthogonal matrix

An *orthogonal matrix* is a square invertible matrix $\mathbf{U}$ such that $\boxed{\mathbf{U}^{-1} = \mathbf{U}^T.}$

If $\mathbf{U}$ is $n \times n$ with orthonormal columns, we have $\mathbf{U}^T\mathbf{U} = \mathbf{I}_n$ and $\mathbf{U}\mathbf{U}^T = \mathbf{I}_n$ (since the left inverse is the right inverse).

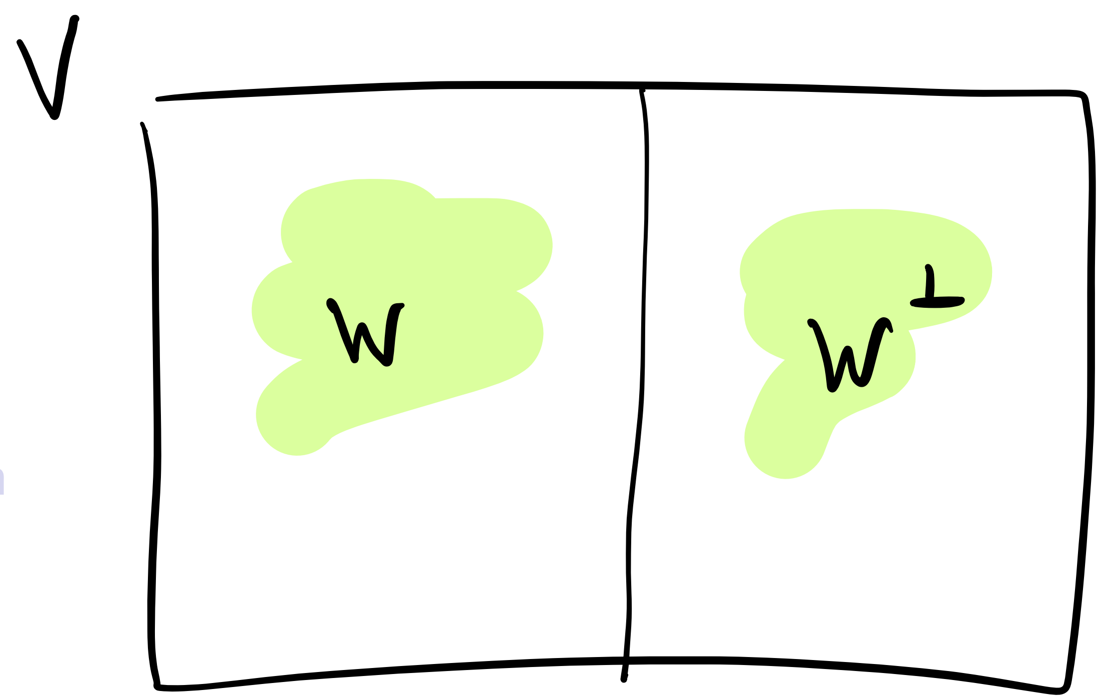
So square matrices with orthonormal columns are called *orthogonal matrices*.

$\underset{\sim}{u}, \underset{\sim}{v}$ orthogonal means $\underset{\sim}{u} \cdot \underset{\sim}{v} = 0$

1. Orthogonal matrices

$\{\underset{\sim}{u}_1, \dots, \underset{\sim}{u}_p\}$ an orth. set means $\underset{\sim}{u}_j \cdot u_k = 0$, $j \neq k$.

2. **Orthogonal subspaces**



3. Orthogonal projections

4. Gram-Schmidt orthogonalization

Orthogonal complement

Let W be a subspace of $\mathbb{R}^n$.

- If $\mathbf{z} \cdot \mathbf{x} = 0$ for all $\mathbf{x} \in W$, we say $\mathbf{z}$ is *orthogonal to W*.
- The *orthogonal complement* of W is the set of all such vectors $\mathbf{z}$.

$$W^\perp = \{ \underset{\sim}{z} \in \mathbb{R}^n : \underset{\sim}{z} \cdot \underset{\sim}{x} = 0 \text{ for all } \underset{\sim}{x} \in W \}$$

Denote the orthogonal complement of a subspace W as $W^\perp$.

\perp

perpendicular

Theorem (Results about orthogonal complements)

Let W be a subspace of $\mathbb{R}^n$. Then:

1. *A vector $\mathbf{x}$ is in $W^\perp$ iff $\mathbf{x}$ is orthogonal to every vector in a set that spans W.*
2. *$W^\perp$ is a subspace of $\mathbb{R}^n$.*

Prove the result.

① Suppose $W = \text{Span}\{\underset{\sim}{w}_1, \dots, \underset{\sim}{w}_p\}$

Claim: $\underset{\sim}{x} \in W^\perp \iff x \cdot \underset{\sim}{w}_j = 0$ for $j = 1, \dots, p$.

proof: "$\Rightarrow$" let $x \in W^\perp$.

Then $\underset{\sim}{x} \cdot \underset{\sim}{w} = 0 \quad \forall \ \underset{\sim}{w} \in W$.

$\Rightarrow \underset{\sim}{x} \cdot \underset{\sim}{w}_j = 0$, $j = 1, \dots, p$, since $\underset{\sim}{w}_1, \dots, w_p \in W$.

"$\Leftarrow$" Suppose $\underset{\sim}{x} \cdot \underset{\sim}{w}_j = 0$ for $j = 1, \dots, p$.

Now, take a vector $\underset{\sim}{y} \in W$.

Then $\underset{\sim}{y} = c_1 \underset{\sim}{w}_1 + \dots + c_p \underset{\sim}{w}_p$ for some $c_1, \dots, c_p$.

So

$$\underset{\sim}{x} \cdot \underset{\sim}{y} = \underset{\sim}{x} \cdot (c_1 \underset{\sim}{w}_1 + \dots + c_p \underset{\sim}{w}_p)$$

$$= c_1 \underbrace{\underset{\sim}{x} \cdot \underset{\sim}{w}_1}_{0} + \dots + c_p \underbrace{\underset{\sim}{x} \cdot \underset{\sim}{w}_p}_{0}$$

$$= 0.$$

So $\underset{\sim}{x}$ is orthogonal to every $\underset{\sim}{y} \in W$.

$$\boxed{W \cap W^\perp = \{\underset{\sim}{0}\}}$$

② $W^\perp$ is a subspace of $\mathbb{R}^n$.

(i) $\underset{\sim}{0} \cdot \underset{\sim}{w} = 0$ for all $\underset{\sim}{w} \in W$.

(ii) $\underset{\sim}{v}_1, \underset{\sim}{v}_2 \in W^\perp$, take $\underset{\sim}{w} \in W$. Then $(\underset{\sim}{v}_1 + \underset{\sim}{v}_2) \cdot \underset{\sim}{w}$

$= \underset{\sim}{v}_1 \cdot \underset{\sim}{w} + \underset{\sim}{v}_2 \cdot \underset{\sim}{w}$

$= 0 + 0$

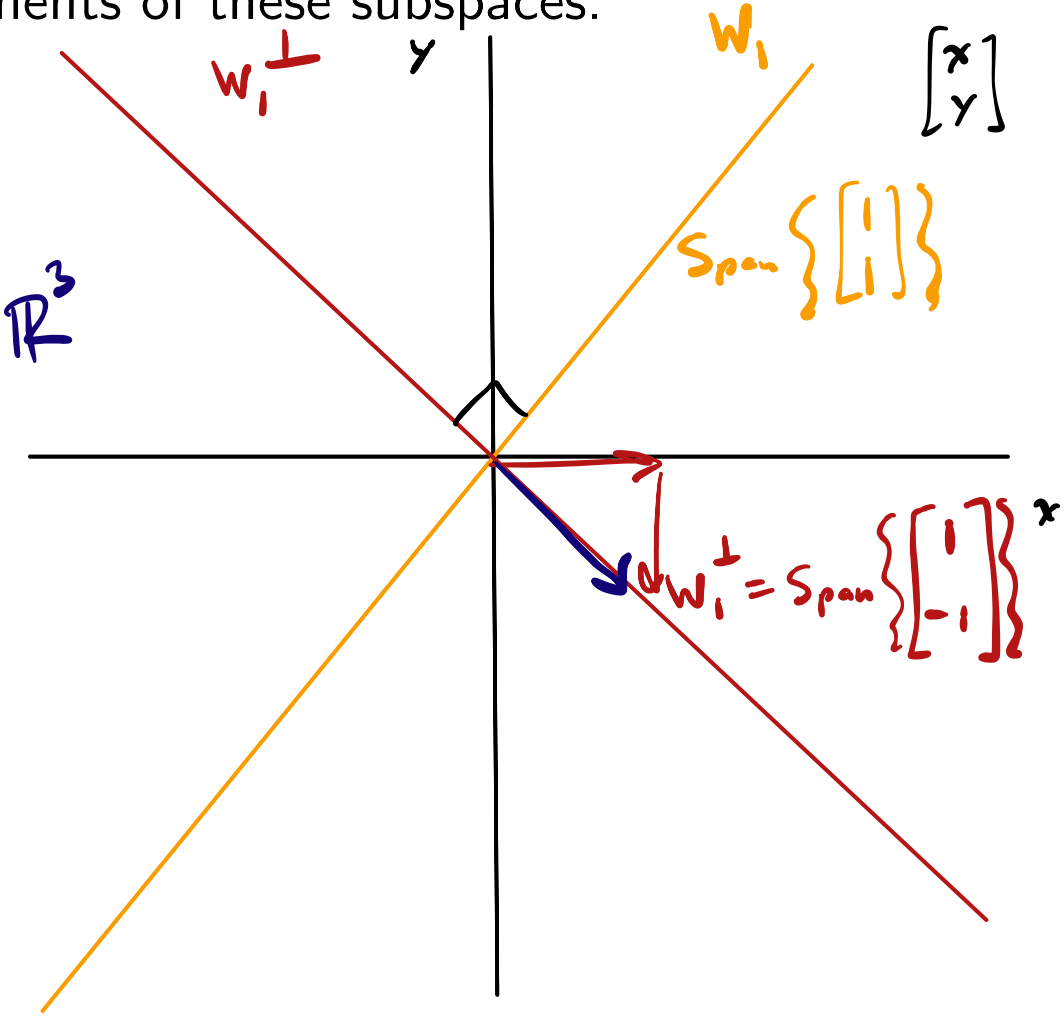
$\Rightarrow \underset{\sim}{v}_1 + \underset{\sim}{v}_2 \in W^\perp.$

(iii) O.Y.O.

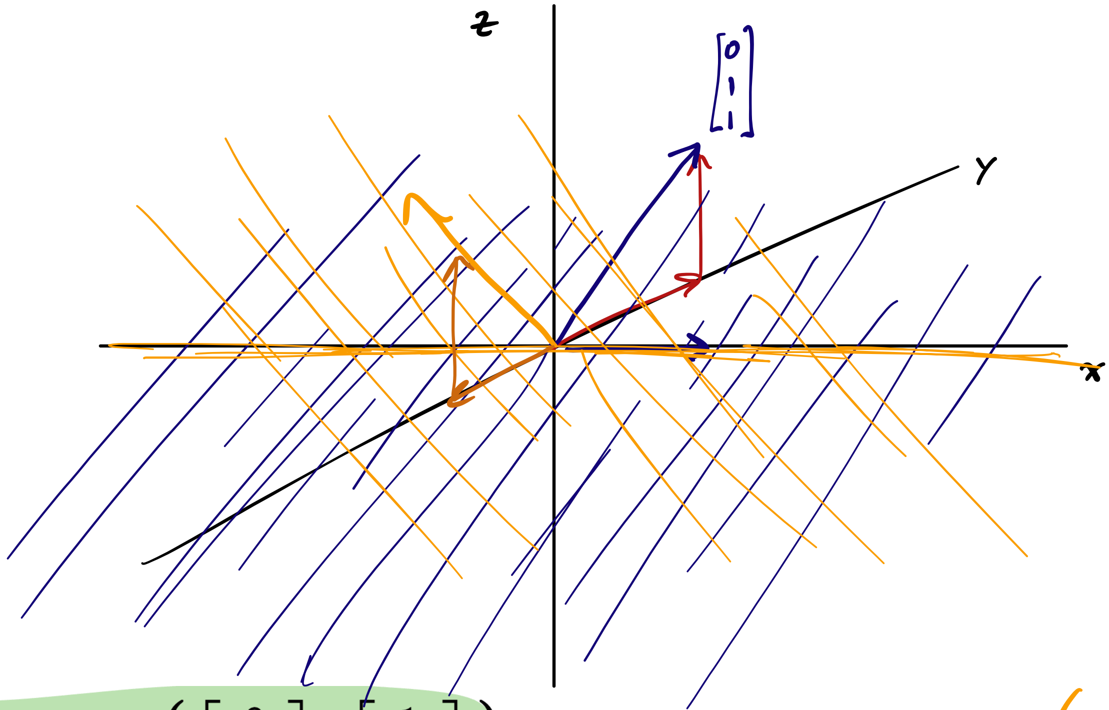
Exercise: Find the orthogonal complements of these subspaces:

1. $W_1 = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \subset \mathbb{R}^2$
2. $W_2 = \text{Span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} \subset \mathbb{R}^3$

Draw pictures.



$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$



2. $$W_2 = \text{Span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$W_2^{\perp} = \text{Span}\left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$$

$W = \text{Col } A$

$W^{\perp} = \text{Nul } A^T$

The next result tells us how to find an orthogonal complement.

Theorem (Orthogonal complement of column and row space)

Let $\mathbf{A}$ *be an* $m \times n$ *matrix. Then* $\boxed{(\text{Col } \mathbf{A})^{\perp} = \text{Nul } \mathbf{A}^T}$ *and* $\boxed{(\text{Row } \mathbf{A})^{\perp} = \text{Nul } \mathbf{A}.}$

Apply first result to A^T.

Prove the result.

$$(\text{Col } A)^{\perp} = \{ \underset{\sim}{x} : \underset{\sim}{x} \cdot \underset{\sim}{a} = 0 \ \forall \ \underset{\sim}{a} \in \text{Col } A \}$$

Claim: $(\text{Col } A)^{\perp} = \text{Nul } A^{T}$

"$\subseteq$" Let $\underset{\sim}{x} \in (\text{Col } A)^{\perp}$.

$\Rightarrow$ $\underset{\sim}{x}$ is orthogonal to every column in A.

Let $A = [\underset{\sim}{a}_1 \cdots \underset{\sim}{a}_n]$

Then $\underset{\sim}{a}_1 \cdot \underset{\sim}{x} = 0, \ldots, \underset{\sim}{a}_n \cdot \underset{\sim}{x} = 0$.

So

$$A^{T}\underset{\sim}{x} = \begin{bmatrix} \underset{\sim}{a}_1^{T} \\ \vdots \\ \underset{\sim}{a}_n^{T} \end{bmatrix} \underset{\sim}{x} = \begin{bmatrix} \underset{\sim}{a}_1^{T}\underset{\sim}{x} \\ \vdots \\ \underset{\sim}{a}_n^{T}\underset{\sim}{x} \end{bmatrix} = \underset{\sim}{0}$$

$\Rightarrow$ $A^{T}\underset{\sim}{x} = \underset{\sim}{0}$ $\Rightarrow$ $\underset{\sim}{x} \in \text{Nul } A$.

"$\supseteq$" Let $x \in \text{Nul } A^{T}$.

$\Rightarrow$ $A^{T}\underset{\sim}{x} = 0$

$\Rightarrow$ $\underset{\sim}{x}$ is orthogonal to every column in A.

$\Rightarrow$ $\underset{\sim}{x} \in (\text{Col } A)^{\perp}$.

Exercise: Find the orthogonal complement of Span $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Find $(\text{Col } A)^{\perp} = \text{Nul } A^T = \{ \underset{\sim}{x} : A^T \underset{\sim}{x} = \underset{\sim}{0} \}.$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\Rightarrow \quad x_1 + x_2 = 0$$

$$x_3 = 0$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

$$\left(\mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\} \right)^{\perp} = \mathrm{Span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Theorem (Chopping result. Cf. Monahan (2008) Result A.6)

For two vector spaces W and V, $W \subset V$ implies $V^\perp \subset W^\perp$.
As a consequence, we have additionally $W = V \iff W^\perp = V^\perp$.

$\subset$ $\supset$ $\qquad$ $\supset$ $\subset$

Prove the result.

Claim: $W \subset V \Rightarrow V^\perp \subset W^\perp$.

Proof: Suppose $W \subset V$. Let $\underset{\sim}{x} \in V^\perp$.

Then $\underset{\sim}{x} \cdot \underset{\sim}{v} = 0$ for all $\underset{\sim}{v} \in V$.

Take any $\underset{\sim}{w} \in W$. Then $\underset{\sim}{w} \in V$, so $\underset{\sim}{x} \cdot \underset{\sim}{w} = 0$.

$$\Rightarrow \quad \underset{\sim}{x} \in W^{\perp}.$$

Theorem (Lemma 2.1 and Results 2.2 and 2.4 of Monahan)

For an $n \times p$ matrix $\mathbf{X}$, we have

1. $\operatorname{Nul} \mathbf{X}^T\mathbf{X} = \operatorname{Nul} \mathbf{X}$
2. $\operatorname{Col} \mathbf{X}^T\mathbf{X} = \operatorname{Col} \mathbf{X}^T$
3. $\mathbf{X}^T\mathbf{X}\mathbf{A} = \mathbf{X}^T\mathbf{X}\mathbf{B} \iff \mathbf{X}\mathbf{A} = \mathbf{X}\mathbf{B}$.

Prove the results.

(1) Claim: $\operatorname{Nul} X^TX = \operatorname{Nul} X$

Proof: "$\subseteq$" $\quad \underset{\sim}{x} \in \operatorname{Nul} X^TX \implies X^TX\underset{\sim}{x} = \underset{\sim}{0}$

$\implies \underset{\sim}{x}^T X^T X \underset{\sim}{x} = 0$

$$\Rightarrow \quad (X\underset{\sim}{x})^T X\underset{\sim}{x} = 0$$

$$\Rightarrow \quad \| X\underset{\sim}{x} \|^2 = 0$$

$$\Rightarrow \quad X\underset{\sim}{x} = \underset{\sim}{0}$$

$$\Rightarrow \quad \underset{\sim}{x} \in \text{Nul } X$$

"$\supseteq$" $\quad \underset{\sim}{x} \in \text{Nul } X \quad \Rightarrow \quad X\underset{\sim}{x} = \underset{\sim}{0}$

$$\Rightarrow \quad X^T X \underset{\sim}{x} = \underset{\sim}{0}$$

$$\Rightarrow \quad \underset{\sim}{x} \in \text{Nul } X^T X.$$

$$(X^T X)^T = X^T X$$

(2) **Claim:** $\text{Col } X^T X = \text{Col } X^T$

Proof: $(\text{Col } X^T X)^\perp = \text{Nul } (X^T X)^T = \text{Nul } X^T X = \text{Nul } X$

$$(\text{Col } X^T)^\perp = \text{Nul } X$$

We have $(\text{Col } X^T X)^\perp = (\text{Col } X^T)^\perp$

$2 \Rightarrow$

$$\text{Col } X^T X = \text{Col } X^T.$$

"Cool Result"

(3) **Claim:** $X^T X A = X^T X B \iff XA = XB$

Proof: "$\Leftarrow$" Suppose $XA = XB$

Then $X^T X A = X^T X B$.

"$\Rightarrow$" Suppose $X^T X A = X^T X B$

$\Rightarrow$ $X^T X A - X^T X B = 0$

$\Rightarrow$ $X^T X (A - B) = 0$

$\Rightarrow$ Columns of $A-B$ are in Nul $X^T X$.

$\Rightarrow$ Columns of $A-B$ are in Nul X.

$\Rightarrow$ $X(A-B) = 0$

$\Rightarrow$ $XA = XB$.

Theorem (An orthogonal set of nonzero vectors is a basis)

If $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ *is an orthogonal set of nonzero vectors in* $\mathbb{R}^n$*, then it is linearly independent and therefore a basis for* $\text{Span}\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$.

An *orthogonal basis* is a basis which is an orthogonal set.

Prove the result.

Claim: $\{\underset{\sim}{u}_1, \ldots, \underset{\sim}{u}_p\}$ orth, nonzero $\Rightarrow$ $\{\underset{\sim}{u}_1, \ldots, \underset{\sim}{u}_p\}$ lin. indep.

Proof: Suppose $c_1 \underset{\sim}{u}_1 + \cdots + c_p \underset{\sim}{u}_p = \underset{\sim}{0}$ for some $c_1, \ldots, c_p$.

Take inner product on both sides:

$$\underset{\sim}{u}_j \cdot \left(c_1 \underset{\sim}{u}_1 + \dots + c_p \underset{\sim}{u}_p \right) = \underset{\sim}{u}_j \cdot \underset{\sim}{0}$$

$$\Leftrightarrow \quad c_j \underbrace{\underset{\sim}{u}_j \cdot \underset{\sim}{u}_j}_{\neq 0} = 0$$

$$\Rightarrow \quad c_j = 0 .$$

Do this for all $j = 1, \dots, p$, we see

$$c_1 = \dots = c_p = 0.$$

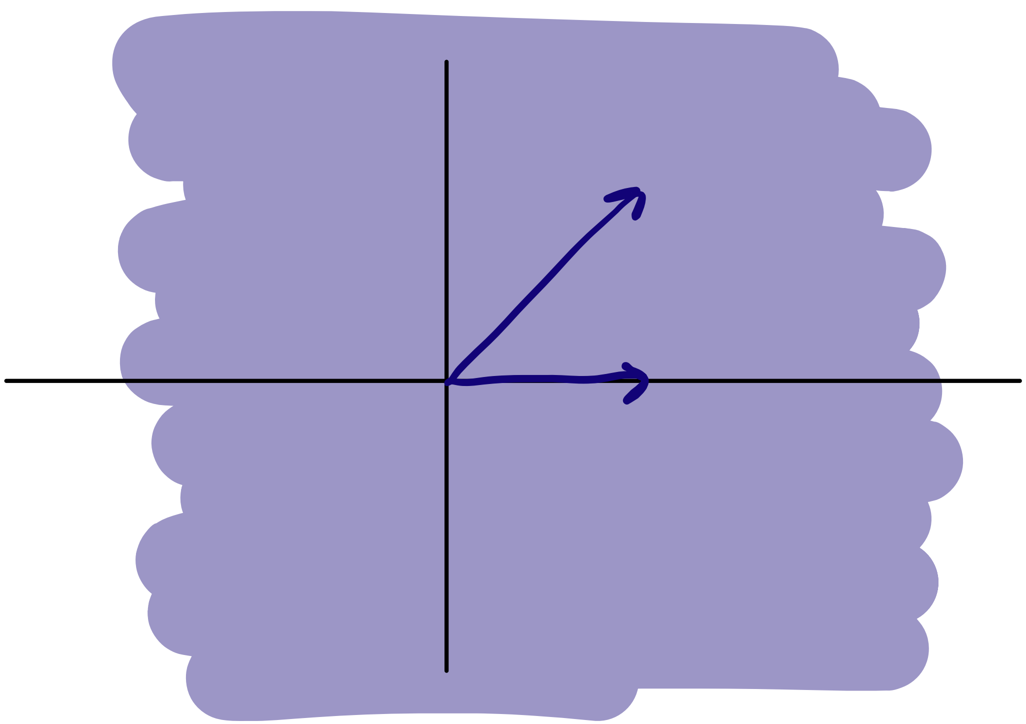
$\Rightarrow \quad \{ \underset{\sim}{u}_1, \dots, \underset{\sim}{u}_p \}$ lin. indep.

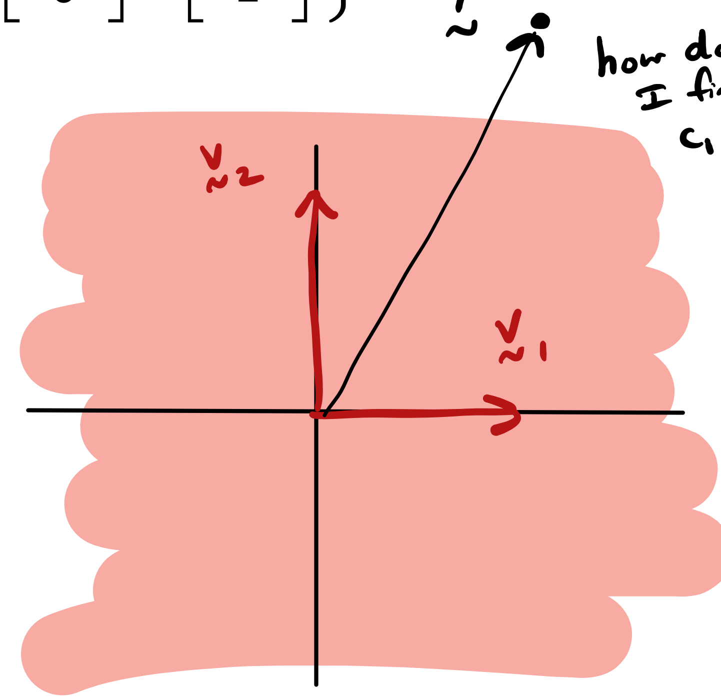
Example: Here are two bases for the same space—one orthogonal, one not:

$$\mathcal{B}_1 = \cancel{\text{span}}\left\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\0\end{bmatrix}\right\} \quad \mathcal{B}_2 = \cancel{\text{span}}\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right\}$$

$\underset{\sim}{y} = c_1\underset{\sim}{v}_1 + c_2\underset{\sim}{v}_2$

how do I find c_1, c_2?





How to find a vector's "coordinates" with respect to an orthogonal basis:

Result (Find a vector's coordinates wrt an orthogonal basis)

Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ *be an orth. basis for a subspace* W *of* $\mathbb{R}^n$*. Then for each* $\mathbf{y} \in W$,

$$\mathbf{y} = c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p, \quad \text{where } c_j = \frac{\mathbf{y} \cdot \mathbf{u}_j}{\mathbf{u}_j \cdot \mathbf{u}_j}, \quad j = 1, \ldots, p.$$

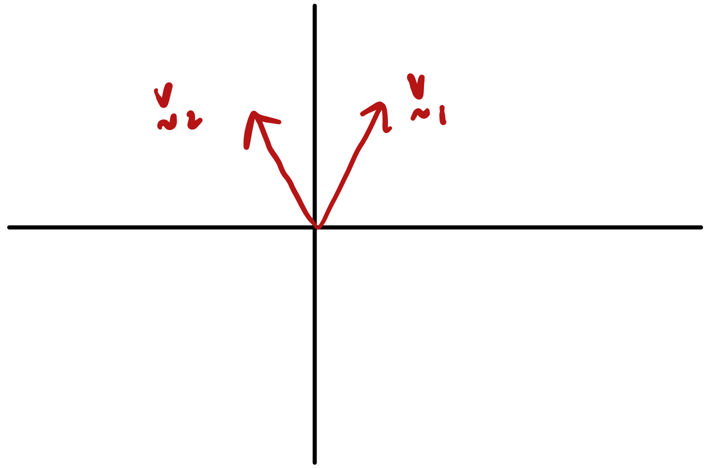
Prove the result.

Write $\quad \underset{\sim}{y} = c_1 \underset{\sim}{u}_1 + \cdots + c_p \underset{\sim}{u}_p.$

Now take inner product of $\underset{\sim}{u}_j$ with both sides:

$$\underset{\sim}{u}_j \cdot \underset{\sim}{y} = \underset{\sim}{u}_j \cdot (c_1 \underset{\sim}{u}_1 + \cdots + c_p \underset{\sim}{u}_p) = c_j \underset{\sim}{u}_j \cdot \underset{\sim}{u}_j$$

$$\iff \qquad c_j = \frac{\underset{\sim}{u}_j \cdot \underset{\sim}{y}}{\underset{\sim}{u}_j \cdot \underset{\sim}{u}_j}$$



$$\begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \sqrt{3}/2 + 1$$

$$\begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = -1/2 + \sqrt{3}$$

Exercise: Find the coefficients to construct $\mathbf{y} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ from the orthogonal basis

$$\left\{ \underset{\underset{\sim}{v}_1}{\begin{bmatrix} \sqrt{3}/2 \\ 1/2 \end{bmatrix}}, \underset{\underset{\sim}{v}_2}{\begin{bmatrix} -1/2 \\ \sqrt{3}/2 \end{bmatrix}} \right\}$$

$$\underset{\sim}{y} = c_1 \underset{\sim}{v}_1 + c_2 \underset{\sim}{v}_2 = \frac{\underset{\sim}{v}_1 \cdot \underset{\sim}{y}}{\underset{\sim}{v}_1 \cdot \underset{\sim}{v}_1} \underset{\sim}{v}_1 + \frac{\underset{\sim}{v}_2 \cdot \underset{\sim}{y}}{\underset{\sim}{v}_2 \cdot \underset{\sim}{v}_2} \cdot \underset{\sim}{v}_2$$

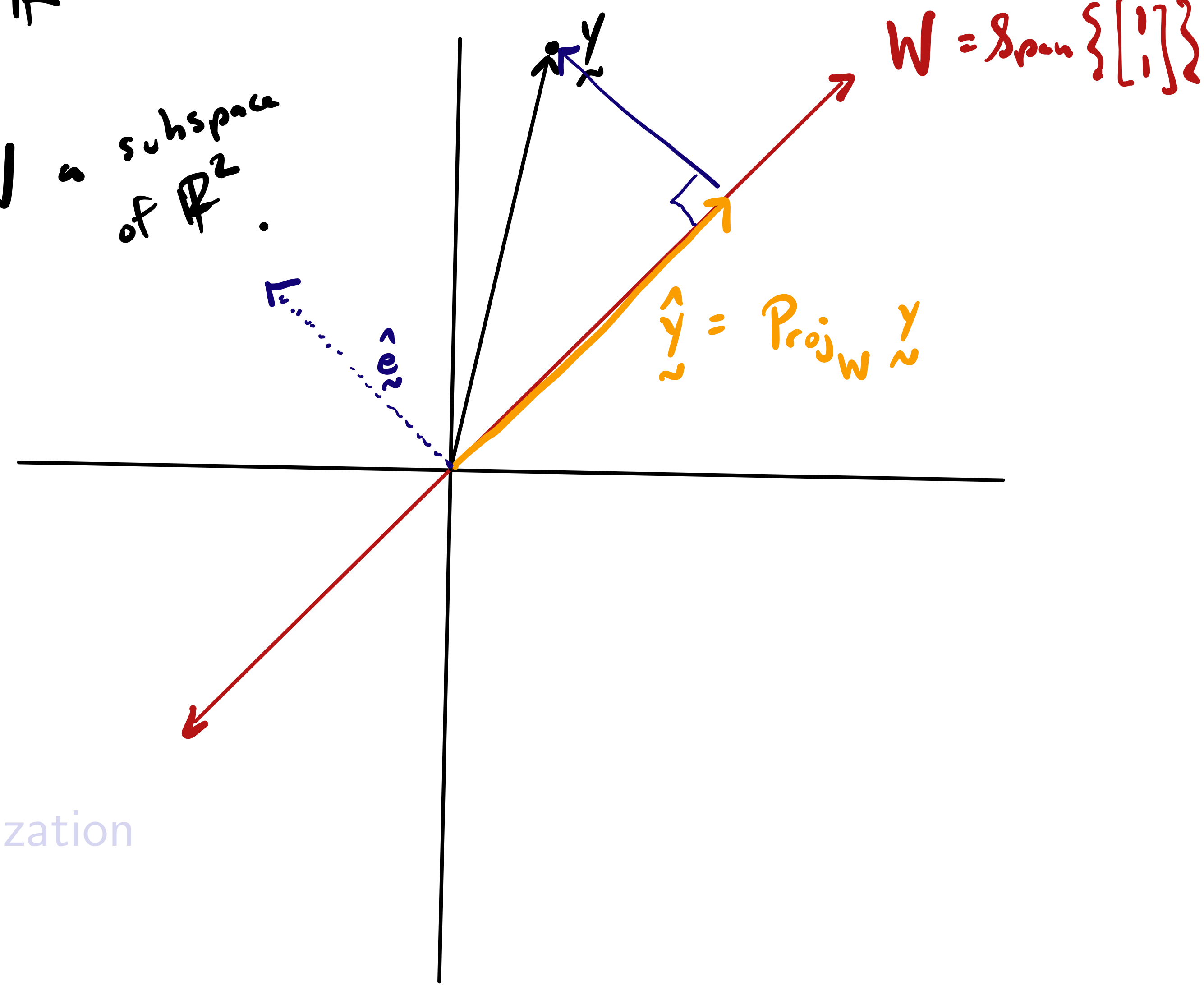
$$\frac{\underset{\sim}{v}_1 \cdot \underset{\sim}{y}}{\underset{\sim}{v}_1 \cdot \underset{\sim}{v}_1} = \left(\frac{\sqrt{3}/2 + 1}{1} \right) = \frac{\sqrt{3}}{2} + 1 \qquad \frac{\underset{\sim}{v}_2 \cdot \underset{\sim}{y}}{\underset{\sim}{v}_2 \cdot \underset{\sim}{v}_2} = \left(\frac{-1/2 + \sqrt{3}}{1} \right) = \sqrt{3} - 1/2$$

$$\left(\frac{\sqrt{3}}{2}+1\right)\begin{bmatrix}\sqrt{3}/2\\ 1/2\end{bmatrix}+\left(\sqrt{3}-\frac{1}{2}\right)\begin{bmatrix}-1/2\\ \sqrt{3}/2\end{bmatrix}$$

$$=\begin{bmatrix}\frac{3}{4}+\frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2}+\frac{1}{4}\\ \frac{\sqrt{3}}{4}+\frac{1}{2} & +\frac{3}{2}-\frac{\sqrt{3}}{4}\end{bmatrix}=\begin{bmatrix}1\\ 2\end{bmatrix}$$

$\underset{\sim}{y} \in \mathbb{R}^2$

W a subspace of $\mathbb{R}^2$.



Theorem (Orthogonal decomposition theorem)

Let W be a subspace of $\mathbb{R}^n$. Then we can decompose any $\mathbf{y} \in \mathbb{R}^n$ $\boxed{\textit{uniquely}}$ *as*

$$\mathbf{y} = \hat{\mathbf{y}} + \hat{\mathbf{e}}, \quad \text{where} \quad \boxed{\hat{\mathbf{y}} \in W} \text{ and } \boxed{\hat{\mathbf{e}} \in W^{\perp}.}$$

Moreover, for any orthogonal basis $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ for W, we have

$$\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \cdots + \frac{\mathbf{y} \cdot \mathbf{u}_p}{\mathbf{u}_p \cdot \mathbf{u}_p}\mathbf{u}_p \quad \text{and} \quad \hat{\mathbf{e}} = \mathbf{y} - \hat{\mathbf{y}}.$$

The vector $\hat{\mathbf{y}}$ is called the *orthogonal projection of $\mathbf{y}$ onto W*, denoted by $\text{proj}_W \mathbf{y}$.

Note that if $\mathbf{y} \in W$ then $\text{proj}_W \mathbf{y} = \mathbf{y}$.

Prove the theorem.

Let $\{\underset{\sim}{u}_1, \dots, \underset{\sim}{u}_p\}$ be an orth. basis for W.

Then $\quad \hat{\underset{\sim}{y}} = \left(\dfrac{\underset{\sim}{y} \cdot \underset{\sim}{u}_1}{\underset{\sim}{u}_1 \cdot \underset{\sim}{u}_1}\right) \underset{\sim}{u}_1 + \cdots + \left(\dfrac{\underset{\sim}{y} \cdot \underset{\sim}{u}_p}{\underset{\sim}{u}_p \cdot \underset{\sim}{u}_p}\right) \underset{\sim}{u}_p \quad \in W$

Also if $\hat{\underset{\sim}{e}} = \underset{\sim}{y} - \hat{\underset{\sim}{y}}$, then we can write

$$\begin{aligned}
\underset{\sim}{u}_j \cdot \hat{\underset{\sim}{e}} &= \underset{\sim}{u}_j \cdot \left(\underset{\sim}{y} - \hat{\underset{\sim}{y}}\right) \\
&= \underset{\sim}{u}_j \cdot \underset{\sim}{y} - \underset{\sim}{u}_j \cdot \left(\left(\dfrac{\underset{\sim}{y} \cdot \underset{\sim}{u}_j}{\underset{\sim}{u}_j \cdot \underset{\sim}{u}_j}\right) \underset{\sim}{u}_j \right) \\
&= \underset{\sim}{u}_j \cdot \underset{\sim}{y} - \underset{\sim}{u}_j \cdot \underset{\sim}{y} \\
&= 0
\end{aligned}$$

for all $j = 1, \ldots, p$. So $\hat{\underset{\sim}{e}} \in W^\perp$.

Uniqueness :

Suppose $\underset{\sim}{y} = \hat{\underset{\sim}{y}}_1 + \hat{\underset{\sim}{e}}_1$ and $\underset{\sim}{y} = \hat{\underset{\sim}{y}}_2 + \hat{\underset{\sim}{e}}_2$,

where $\hat{\underset{\sim}{y}}_1, \hat{\underset{\sim}{y}}_2 \in W$ and $\hat{\underset{\sim}{e}}_1, \hat{\underset{\sim}{e}}_2 \in W^\perp$.

Then $\quad \hat{\underset{\sim}{y}}_1 + \hat{\underset{\sim}{e}}_1 = \hat{\underset{\sim}{y}}_2 + \hat{\underset{\sim}{e}}_2$

$$\iff \underbrace{\hat{\underset{\sim}{y}}_1 - \hat{\underset{\sim}{y}}_2}_{\in W} = \underbrace{\hat{\underset{\sim}{e}}_2 - \hat{\underset{\sim}{e}}_1}_{\in W^\perp} \implies \begin{aligned} \hat{\underset{\sim}{y}}_1 - \hat{\underset{\sim}{y}}_2 &= \underset{\sim}{0} \\ \hat{\underset{\sim}{e}}_2 - \hat{\underset{\sim}{e}}_1 &= \underset{\sim}{0} \end{aligned}$$

$W \cap W^\perp = \{\mathbf{0}\}$

$\Rightarrow \hat{\mathbf{y}}_1 = \hat{\mathbf{y}}_2$

$\hat{\mathbf{e}}_1 = \hat{\mathbf{e}}_2$.

Theorem (Best approximation theorem)

Let W be a subspace of $\mathbb{R}^n$, $\mathbf{y}$ any vector in $\mathbb{R}^n$, and $\hat{\mathbf{y}} = \text{proj}_W \mathbf{y}$. Then

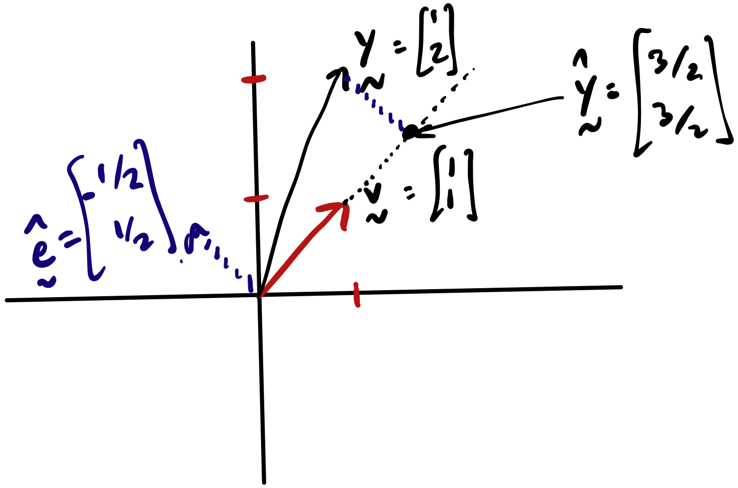
$$\|\mathbf{y} - \hat{\mathbf{y}}\| < \|\mathbf{y} - \mathbf{v}\| \quad \textit{for all } \mathbf{v} \in W \textit{ distinct from } \hat{\mathbf{y}}.$$

Prove the result.

$$\|\mathbf{y} - \mathbf{v}\|^2 = \|(\mathbf{y} - \hat{\mathbf{y}}) + (\hat{\mathbf{y}} - \mathbf{v})\|^2$$

$$= \|\mathbf{y} - \hat{\mathbf{y}}\|^2 + \underbrace{\|\hat{\mathbf{y}} - \mathbf{v}\|^2}_{>0} + \underbrace{2\,(\overbrace{\mathbf{y} - \hat{\mathbf{y}}}^{\in W^\perp}) \cdot (\overbrace{\hat{\mathbf{y}} - \mathbf{v}}^{\in W})}_{=0}$$

$$\Rightarrow \|\mathbf{y} - \hat{\mathbf{y}}\|^2 < \|\mathbf{y} - \mathbf{v}\|^2.$$



$$\underset{\sim}{y}\cdot\underset{\sim}{v} = \begin{pmatrix}1\\2\end{pmatrix}\cdot\begin{pmatrix}1\\1\end{pmatrix} = 3$$

$$\underset{\sim}{v}\cdot\underset{\sim}{v} = \begin{pmatrix}1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\1\end{pmatrix} = 2$$

Exercise: Let $\mathbf{v} = \begin{bmatrix}1\\1\end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix}1\\2\end{bmatrix}$. Give the orthogonal projection $\hat{\mathbf{y}}$ of $\mathbf{y}$ on Span$\{\mathbf{v}\}$. Draw pictures! Check orthogonality of $\mathbf{y} - \hat{\mathbf{y}}$ and $\mathbf{y}$.

$$\hat{\underset{\sim}{y}} = \frac{\underset{\sim}{y}\cdot\underset{\sim}{v}}{\underset{\sim}{v}\cdot\underset{\sim}{v}}\underset{\sim}{v} = \left(\frac{3}{2}\right)\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}3/2\\3/2\end{bmatrix}$$

$$\begin{bmatrix}3/2\\3/2\end{bmatrix} + \begin{bmatrix}-1/2\\1/2\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}$$

$$\hat{\underset{\sim}{e}} = \underset{\sim}{y} - \hat{\underset{\sim}{y}} = \begin{bmatrix}1\\2\end{bmatrix} - \begin{bmatrix}3/2\\3/2\end{bmatrix} = \begin{bmatrix}-1/2\\1/2\end{bmatrix}$$

The Gram-Schmidt process turns a basis into an orthogonal basis.

The Gram–Schmidt Process

Given a basis $\{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$ for a subspace W of $\mathbb{R}^n$, define

$$\mathbf{v}_1 = \mathbf{x}_1$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \boxed{\frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}}\mathbf{v}_1 = \underset{\sim}{x}_2 - \text{Proj}_{\text{Span}\{\underset{\sim}{v}_1\}} \underset{\sim}{x}_2$$

$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 = \underset{\sim}{x}_3 - \text{Proj}_{\text{Span}\{\underset{\sim}{v}_1, \underset{\sim}{v}_2\}} \underset{\sim}{x}_3$$

$$\vdots$$

$$\mathbf{v}_p = \mathbf{x}_p - \frac{\mathbf{x}_p \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_p \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 - \cdots - \frac{\mathbf{x}_p \cdot \mathbf{v}_{p-1}}{\mathbf{v}_{p-1} \cdot \mathbf{v}_{p-1}}\mathbf{v}_{p-1}$$

Then $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an orthogonal basis for W. In addition

$$\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} = \text{Span}\{\mathbf{x}_1, \ldots, \mathbf{x}_k\} \qquad \text{for } 1 \le k \le p \tag{1}$$

Prove the result.

Exercise: Use the Gram-Schmidt process to orthogonalize the basis

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

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