STAT 714 fa 2025

Linear algebra review 5/6

Eigenvalues and eigenvectors

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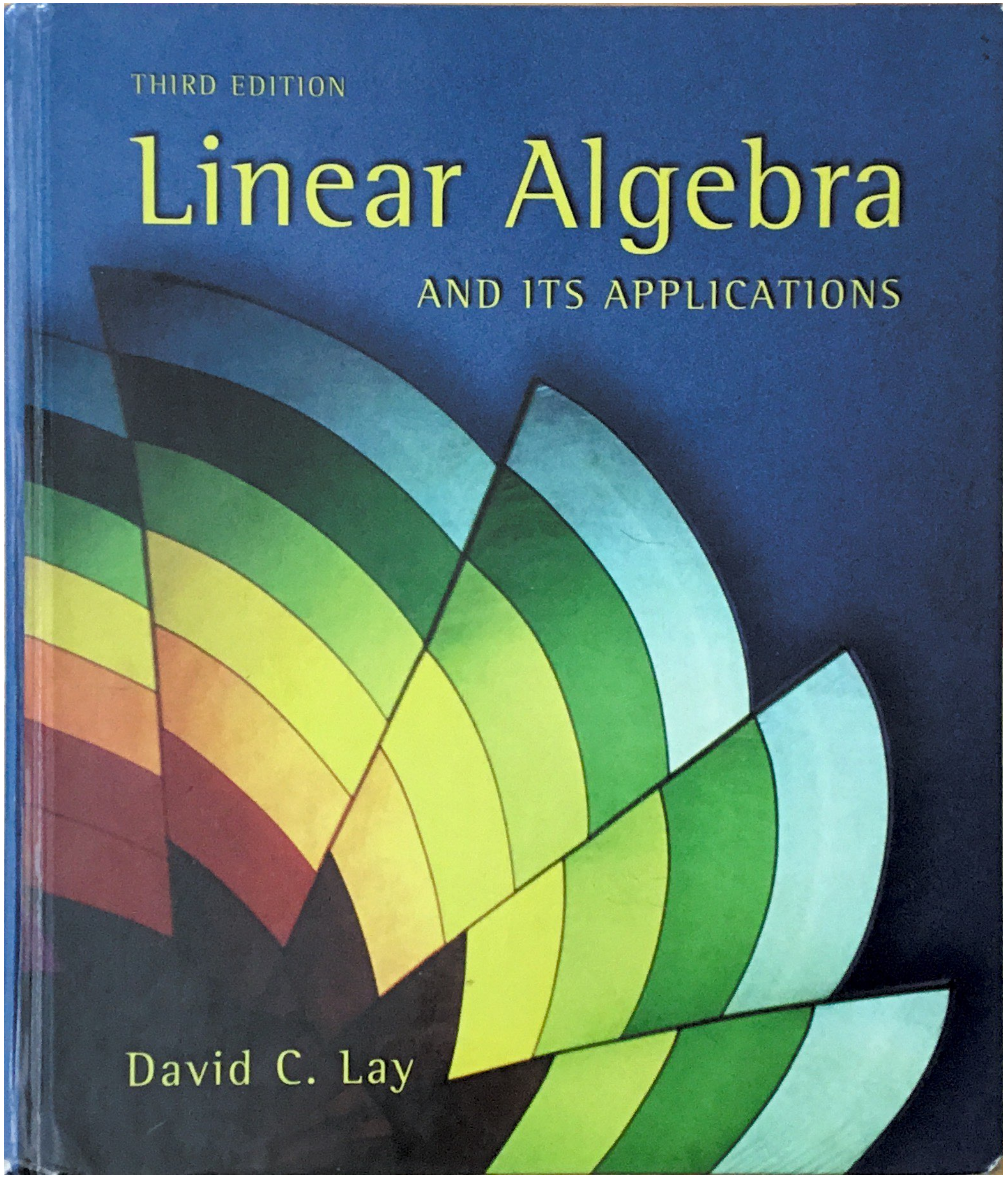
These slides are an instructional aid; their sole purpose is to display, during the lecture, definitions, plots, results, etc. which take too much time to write by hand on the blackboard. They are not intended to explain or expound on any material.

These notes include scanned excerpts from Lay (2003):



Square $\underset{n \times n}{A}$

1. Eigenvalues and eigenvectors
2. Determinants
3. Diagonalization

$$A \underset{\sim}{x} = \lambda \underset{\sim}{x}$$

eigenvector (nonzero)

Eigenvectors and eigenvalues

Let $\mathbf{A}$ be an $\boxed{n \times n}$ matrix.

1. An *eigenvector* of $\mathbf{A}$ is a nonzero vec. $\mathbf{x}$ such that $\mathbf{A}\mathbf{x} = \lambda \mathbf{x}$ for some scalar λ.
2. A scalar λ is an *eigen~~vector~~* value of $\mathbf{A}$ if there is a nontrivial solution to $\mathbf{A}\mathbf{x} = \lambda \mathbf{x}$. Such an $\mathbf{x}$ is called an *eigenvector corresponding to λ*.

Interpretation: The magnitudes of the eigenvalues of $\mathbf{A}$ represent the amount by which $\mathbf{A}$ stretches or shrinks certain vectors.

Exercise: For $\mathbf{A} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$ check whether

1. the vectors $\mathbf{u} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ are eigenvectors.
2. the values -4 and 6 are eigenvalues.

(2) $A\underset{\sim}{u} = \lambda \underset{\sim}{u}$ for some λ?

$\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 14 \\ 14 \end{bmatrix} = 7\begin{bmatrix} 2 \\ 2 \end{bmatrix}$, so yes.

$\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -9 \\ 11 \end{bmatrix}$, not an eigenvector.

② Is -4 an eigenvalue of A?

Is there a nonzero solution to this:

$$A\underset{\sim}{x} = -4\underset{\sim}{x}$$

$$A\underset{\sim}{x} + 4\underset{\sim}{x} = \underset{\sim}{0}$$

$$(A + 4I)\underset{\sim}{x} = \underset{\sim}{0}$$

$$\begin{bmatrix} 5 & 6 \\ 5 & 6 \end{bmatrix} \underset{\sim}{x} = \underset{\sim}{0}$$

write

$$\left[\begin{array}{cc|c} 5 & 6 & 0 \\ 5 & 6 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 5 & 6 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow$ $5x_1 + 6x_2 = 0$ $\Leftrightarrow$ $x_1 = -\frac{6}{5}x_2$

x_2 free $\quad x_2 = x_2$

$$\underset{\sim}{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -6/5 \\ 1 \end{bmatrix}, \quad x_2 \in \mathbb{R}.$$

Yes, -4 is an eigenvalue.

$$A\underset{\sim}{x} = \lambda \underset{\sim}{x} \iff (A - \lambda I)\underset{\sim}{x} = \underset{\sim}{0}$$

Eigenspaces

If λ is an eigenvalue of $\mathbf{A}$, the set of all solutions to $(\mathbf{A} - \lambda \mathbf{I}_n)\mathbf{x} = \mathbf{0}$ is called the *eigenspace* of $\mathbf{A}$ corresponding to λ.

$\lambda = 2$

Exercise: An eigenvalue of the matrix below is $\boxed{2}$. Find a basis for the corresponding eigenspace:

$$\mathbf{A} = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$$

$$(A - \lambda I) = \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix}$$

$\lambda = 2$

$$(A - \lambda I)\underset{\sim}{x} = \underset{\sim}{0}$$

$$\left[\begin{array}{ccc|c} 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & -1 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \quad 2x_1 - x_2 + 6x_3 = 0$$

x_2 free

x_3 free

$$\Leftrightarrow \quad \begin{aligned} x_1 &= -3x_3 + \tfrac{1}{2}x_2 \\ x_2 &= x_2 \\ x_3 &= x_3 \end{aligned}$$

Solutions

$$\left\{ \underset{\sim}{x} : \underset{\sim}{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix},\ x_2, x_3 \in \mathbb{R} \right\}$$

$$= \operatorname{Span}\left\{ \underbrace{\begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}}_{\text{Form a basis for the eigenspace of } \lambda = 2.} \right\}$$

$$= \{ \underset{\sim}{x} : A\underset{\sim}{x} = 2\underset{\sim}{x} \}$$

Theorem (Linear independence of eigenspaces)

If $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are eigenvectors corresponding to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of a matrix, then $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is linearly independent

Prove the result.

Suppose $\{\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_r\}$ is lin. <u>depend</u>. (by way of contradiction)

Then there is some $p < r$ such that

$\{\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_p\}$ $\underline{\underline{\text{is}}}$ lin. independent.

Then suppose I can write

$$\underset{\sim}{v}_{p+1} = c_1 \underset{\sim}{v}_1 + \dots + c_p \underset{\sim}{v}_p \quad \text{for some } c_1, \dots, c_p.$$

Write

$$A \underset{\sim}{v}_{p+1} = \lambda_{p+1} \underset{\sim}{v}_{p+1}$$

$$= \lambda_{p+1} \left(c_1 \underset{\sim}{v}_1 + \dots + c_p \underset{\sim}{v}_p \right)$$

Also

$$A \underset{\sim}{v}_{p+1} = A \left(c_1 \underset{\sim}{v}_1 + \dots + c_p \underset{\sim}{v}_p \right)$$

$$= c_1 A \underset{\sim}{v}_1 + \dots + c_p A \underset{\sim}{v}_p$$

$$= c_1 \lambda_1 \underset{\sim}{v}_1 + \dots + c_p \lambda_p \underset{\sim}{v}_p$$

$\Leftrightarrow$

$$\lambda_{p+1} \left(c_1 \underset{\sim}{v}_1 + \dots + c_p \underset{\sim}{v}_p \right) = c_1 \lambda_1 \underset{\sim}{v}_1 + \dots + c_p \lambda_p \underset{\sim}{v}_p$$

$\Leftrightarrow$

$$c_1 (\lambda_{p+1} - \lambda_1) \underset{\sim}{v}_1 + \dots + c_p (\lambda_{p+1} - \lambda_p) \underset{\sim}{v}_p = 0$$

$$\Leftrightarrow \quad c_1 \underbrace{(\lambda_{p+1} - \lambda_1)}_{\neq 0} = \dots = c_p \underbrace{(\lambda_{p+1} - \lambda_p)}_{\neq 0} = 0$$

because $\{\underset{\sim}{v}_1, \dots, \underset{\sim}{v}_p\}$ is lin indep.

$$\Leftrightarrow \quad c_1 = \dots = c_p = 0 .$$

Recall that $\underset{\sim}{v}_{p+1} = c_1 \underset{\sim}{v}_1 + \dots + c_p \underset{\sim}{v}_p$, so this means

$\underset{\sim}{x}_{p+1} = \underset{\sim}{0}.$

But $\underset{\sim}{x}_{p+1}$ is an eigenvector, so it cannot be $\underset{\sim}{0}$!. $\square$

The Invertible Matrix Theorem (continued)

Let A be an $n \times n$ matrix. Then A is invertible if and only if:

s. $\boxed{\text{The number } 0 \text{ is } \textit{not} \text{ an eigenvalue of } A.}$

t. The determinant of A is *not* zero.

$A\underset{\sim}{x} = 0$

Prove the first of the above results.

$\underline{A \text{ invertible}} \iff 0$ not an eigenvalue

A not invertible $\iff 0$ is an eigenvalue

"$\Leftarrow$" Let 0 be an eigenvalue.

Then there is a nonzero $\underset{\sim}{x}$ such that

$$\boxed{A\underset{\sim}{x} = 0\underset{\sim}{x} = 0}$$

If A invertible $A \sim I$.

So if A invertible, $A\underset{\sim}{x} = \underset{\sim}{0}$ has only the trivial solution, so if $A\underset{\sim}{x} = 0$ for some nonzero $\underset{\sim}{x}$, then A is not invertible.

"$\Rightarrow$" Assume A not invertible.

Show $A x = 0 \cdot \underset{\sim}{x}$ for some nonzero $\underset{\sim}{x}$

has fewer than n pivot colums, so it has more than 1 solution.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$(-1)^{n+1}$

Recall $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$

$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix} =$$

For larger matrices the determinant is defined as follows:

Definition of determinant by co-factor expansion

Let $\mathbf{A}$ be an $n \times n$ matrix with i, j entry a_{ij} and let $\mathbf{A}_{(i,j)}$ be the matrix $\mathbf{A}$ with row i and column j removed. Then, for $i, j = 1, \dots, n$, define the (i, j)-cofactor as

$$C_{ij} = (-1)^{i+j} \det \mathbf{A}_{(i,j)}.$$

Then for any i and j we have $\det \mathbf{A} = \sum_{k=1}^{n} a_{ik} C_{ik} = \sum_{k=1}^{n} a_{kj} C_{kj}$.

Often write $\det \mathbf{A}$ as $|\mathbf{A}|$.

This requires over $n!$ multiplications, so computers use a different method.

$$\underline{a_{11}}\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} - a_{21}\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$+ \cdots + (-1)^{n+1} a_{n1}\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$|A| = -(-2)\begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} = -2$$

(Go across bottom row)

Exercise: Compute the determinant of

$$\mathbf{A} = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & 1 \\ 0 & -2 & 0 \end{bmatrix}. \qquad \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

(go down first column)

$$|A| = 1 \cdot \begin{vmatrix} 4 & -1 \\ -2 & 0 \end{vmatrix} + (-2)\begin{vmatrix} 5 & 0 \\ -2 & 0 \end{vmatrix} + 0 \begin{vmatrix} 5 & 0 \\ 4 & -1 \end{vmatrix}$$

$$= 1(-2) = -2$$

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{vmatrix} d & -c \\ -b & a \end{vmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad \begin{bmatrix} + & - \\ - & + \end{bmatrix}$$

The determinant and cofactors give a formula for a matrix inverse:

Theorem (An inverse formula using cofactors)

If $\mathbf{A}$ *is an invertible* $n \times n$ *matrix, we have*

transpose of cofactor matrix

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \boxed{\begin{bmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \dots & C_{nn} \end{bmatrix}}$$

Exercise: Compute the inverse of the matrix on the previous slide.

Theorem (Some properties of determinants)

Let $\mathbf{A}$ *and* $\mathbf{B}$ *be* $n \times n$ *matrices. We have*

1. $\mathbf{A}$ *is invertible if and only if* $\det \mathbf{A} \neq 0$. ← See from cofactor version of inverse.
2. $\det \mathbf{A}^T = \det \mathbf{A}$
3. $\det \underline{\mathbf{AB}} = (\underline{\det \mathbf{A}})(\underline{\det \mathbf{B}})$

Discuss above results.

Theorem (Finding eigenvalues with the characteristic equation)

A scalar λ is an eigenvalue of an $n \times n$ matrix $\mathbf{A}$ iff λ satisfies $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$.

The equation $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ is called the *characteristic equation*.

RHS an n-degree polynomial, which has n roots (some roots may be complex).

Prove the above result.

Exercise: Find the eigenvalues of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}.$$

Claim: λ an eigenvalue of A

$$\iff \quad |A - \lambda I| = 0.$$

Proof:

$$|A - \lambda I| = 0$$

$\iff$ $A - \lambda I$ is not invertible

$\iff$ $(A - \lambda I)\underset{\sim}{x} = \underset{\sim}{0}$ has a non-trivial solution.

$\iff$ $A\underset{\sim}{x} = \lambda \underset{\sim}{x}$

for some nonzero $\underset{\sim}{x}$.

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}.$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 3 & 3 \\ -3 & -5-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)\begin{vmatrix} -5-\lambda & -3 \\ 3 & 1-\lambda \end{vmatrix}$$

$$-(-3)\begin{vmatrix} 3 & 3 \\ 3 & 1-\lambda \end{vmatrix}$$

$$+\,3 \begin{vmatrix} 3 & 3 \\ -5-\lambda & -3 \end{vmatrix}$$

$$= \cdots\cdots =$$

$$= (1-\lambda)(2+\lambda)^2$$

$$= 0$$

Eigen values are $\lambda = 1$

$\lambda = -2$ (Multiplicity 2)

Theorem (Eigenvalues of a triangular matrix)

For a triangular matrix

1. *the determinant is the product of the entries on the main diagonal.*
2. *the eigenvalues are the entries on the main diagonal.*

Prove the results.

$$\underset{n\times n}{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & & a_{2n} \\ \text{zeros} & & \ddots & \vdots \\ & & & a_{nn} \end{bmatrix} \qquad |A| = \prod_{i=1}^{n} a_{ii}$$

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & & a_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & & & a_{nn} \end{vmatrix}$$

$$= a_{11} \begin{vmatrix} a_{22} & & a_{2n} \\ & \ddots & \vdots \\ & & a_{nn} \end{vmatrix}$$

$$= a_{11} \quad a_{22} \begin{vmatrix} a_{33} & \cdots & a_{3n} \\ & \ddots & \\ & & a_{nn} \end{vmatrix}$$

$$= \cdots$$

$$= a_{11} \quad a_{22} \cdot \ldots \cdot a_{nn}$$

$$|A - \lambda I| = \begin{vmatrix} a_{11} - \lambda & 0 & & 0 \\ & a_{22} - \lambda & & 0 \\ \text{zero} & & \ddots & \\ & & & a_{nn} - \lambda \end{vmatrix} = \prod_{i=1}^{n} (a_{ii} - \lambda)$$

roots are $a_{11}, a_{22}, \ldots, a_{nn}$.

The *trace* of a square matrix $\mathbf{A}$, denoted $\text{tr}(\mathbf{A})$, is the sum of its diagonal entries.

Theorem (Properties of the trace)

For any matrices $\mathbf{A}$ *and* $\mathbf{B}$, *we have*

1. $\text{tr}(\mathbf{AB}) = \text{tr}(\mathbf{BA})$.
2. $\text{tr}(\mathbf{A}^T\mathbf{A}) = \sum_i \sum_j a_{ij}^2$, *where* a_{ij} *are the entries of* $\mathbf{A}$.

Prove in hw 3.

The function $p_{\mathbf{A}}(t) = \det(t\mathbf{I} - \mathbf{A})$ is called the *characteristic polynomial* of $\mathbf{A}$.

Theorem (Expansion of characteristic polynomial)

The characteristic polynomial of an $n \times n$ matrix $\mathbf{A}$ has the terms

$$p_{\mathbf{A}}(t) = t^n - (\operatorname{tr}\mathbf{A})t^{n-1} + \cdots + (-1)^n \det \mathbf{A}.$$

Exercise: For an $n \times n$ matrix $\mathbf{A}$ with eigenvalues $\lambda_1, \ldots, \lambda_n$, use above to show

1. $\operatorname{tr}\mathbf{A} = \sum_{i=1}^n \lambda_i$
2. $\det \mathbf{A} = \prod_{i=1}^n \lambda_i$

Theorem (Further properties of the determinant)

1. $|\mathbf{A}^{-1}| = 1/|\mathbf{A}|$
2. $|c\mathbf{A}| = c^n|\mathbf{A}|$ *if* $\mathbf{A}$ *is* $n \times n$
3. $\left|\begin{array}{cc} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{array}\right| = |\mathbf{A}||\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}| = |\mathbf{D}||\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C}|.$

See Res A.18 of Monahan (2008).

A square matrix $\mathbf{A}$ is *diagonalizable* if $\mathbf{A} = \mathbf{PDP}^{-1}$ with $\mathbf{P}$ invertible, $\mathbf{D}$ diagonal.

Theorem (Sufficient and necessary conditions for diagonalizability)

An $n \times n$ matrix $\mathbf{A}$ can be written $\mathbf{A} = \mathbf{PDP}^{-1}$ with $\mathbf{D}$ diag. and $\mathbf{P}$ invertible iff

1. *the columns of $\mathbf{P}$ are n linearly independent eigenvectors of $\mathbf{A}$, and*
2. *the diagonal entries of $\mathbf{D}$ are the corresponding eigenvalues of $\mathbf{A}$.*

Prove the result.

Exercise: If possible, diagonalize the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}.$$

Steps:

1. Find the eigenvalues of $\mathbf{A}$.
2. Find three linearly indep. eigenvectors (if not possible, $\mathbf{A}$ not diagonalizable).
3. Give (if possible) the diagonalization $\mathbf{A} = \mathbf{PDP}^{-1}$.

Lay, D. C. (2003). *Linear algebra and its applications. Third edition.* Pearson Education.

Monahan, J. F. (2008). *A primer on linear models*. CRC Press.