STAT 714 fa 2025

Linear algebra review 6/6

Symmetric matrices and quadratic forms

Symmetric

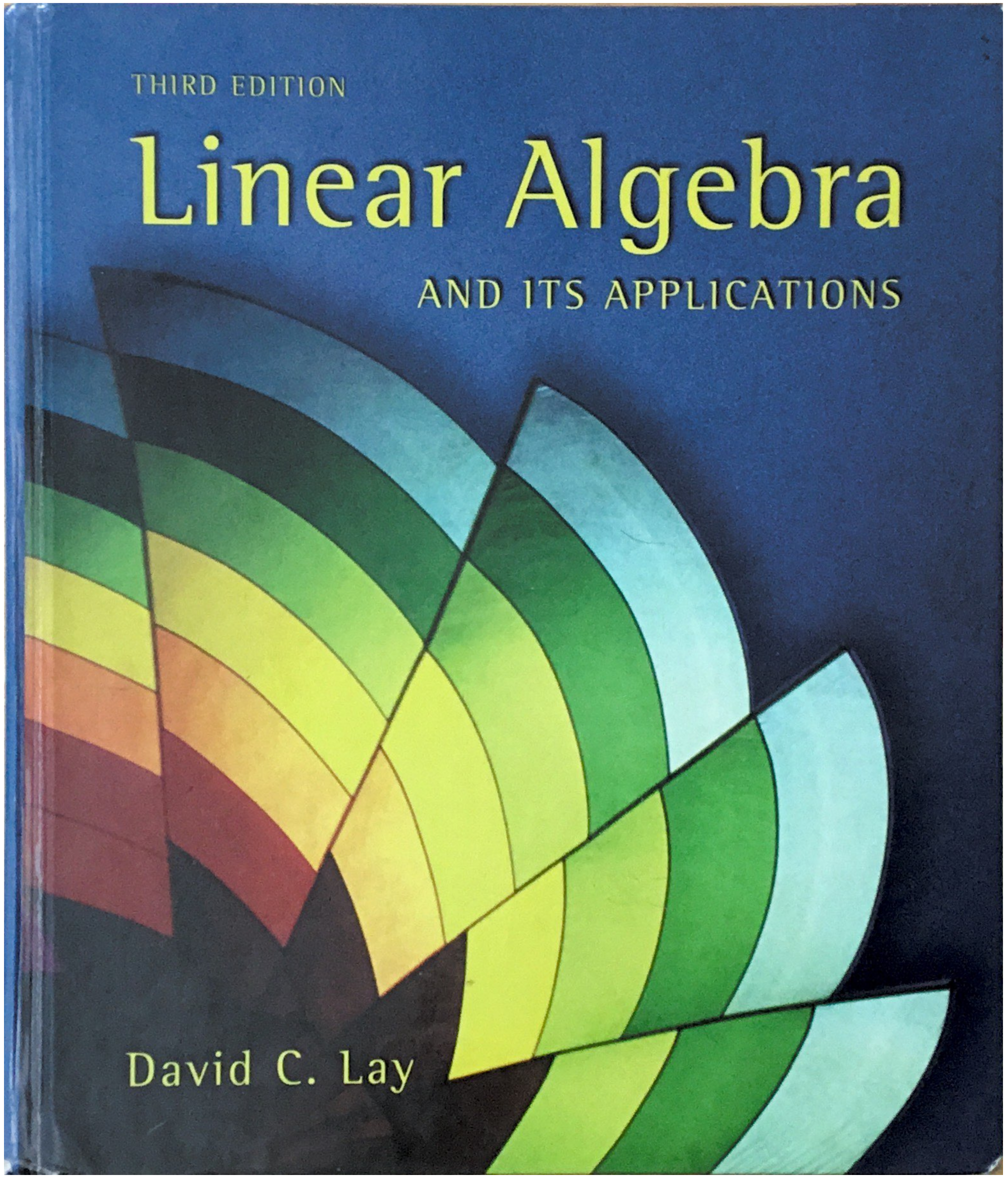
$$\underset{n\times n}{A} = A^T$$

Karl B. Gregory

University of South Carolina

These slides are an instructional aid; their sole purpose is to display, during the lecture, definitions, plots, results, etc. which take too much time to write by hand on the blackboard. They are not intended to explain or expound on any material.

These notes include scanned excerpts from Lay (2003):



Diagonalization

$$A = P D P^T$$

1. Spectral decomposition

$$A = P D P^{-1}$$

symm

2. Quadratic forms

3. Singular value decomposition

Some definitions for symmetric matrices

- A *symmetric* matrix $\mathbf{A}$ is a matrix such that $\mathbf{A} = \mathbf{A}^T$ (necessarily square).
- The set of eigenvalues of a symmetric matrix $\mathbf{A}$ is called the *spectrum* of $\mathbf{A}$.

If $\underset{n\times n}{A}$ is symm. with eigenvectors $\underset{\sim}{u}_1, \dots, \underset{\sim}{u}_n$

and eigenvalues $\lambda_1, \dots, \lambda_n$, we can write

$$A = \sum_{j=1}^{n} \lambda_j \underset{\sim}{u}_j \underset{\sim}{u}_j^T = U \Lambda U^T \dots\dots$$

$\det(A - \lambda I) = 0$

Theorem (Spectral theorem for symmetric matrices)

Let $\mathbf{A}$ *be an* $n \times n$ *symmetric matrix. Then:*

1. *The eigenvalues of* $\mathbf{A}$ *are all real-valued.*
2. *For each eigenvalue, the dimension of the corresponding eigenspace is equal to the multiplicity of the eigenvalue as a root of the characteristic polynomial.*
3. *Eigenspaces corresponding to different (distinct) eigenvalues are orthogonal.*
4. $\mathbf{A}$ *is orthogonally diagonalizable, i.e. we can write* $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^T = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$*, where* $\mathbf{P}$ *is an orthogonal matrix and* $\mathbf{D}$ *is a diagonal matrix.*

Prove 1 and 3. Number 4 can be shown with Schur factorization).

(1) $\underset{n\times n}{A}$ is symm. => All eigenvalues are real-valued.

<u>proof</u>: Let $\lambda \in \mathbb{C}$ be a possibly complex eigenvalue.

Then if $\underset{\sim}{x}$ is an eigenvector corresponding to λ, it may have complex-valued entries.

So write $\underset{\sim}{x} = \underset{\sim}{a} + i\underset{\sim}{b}$, $\underset{\sim}{a}, \underset{\sim}{b} \in \mathbb{R}^n$

We have $A\underset{\sim}{x} = \lambda \underset{\sim}{x}$.

$z \in \mathbb{C}$, $z = a + ib$ $a, b \in \mathbb{R}.$ $i = \sqrt{-1}$ $\bar{z} = a - ib$

Write

$$\bar{\underset{\sim}{x}}^T A \underset{\sim}{x} = (\underset{\sim}{a} - i\underset{\sim}{b})^T A (\underset{\sim}{a} + i\underset{\sim}{b}) \qquad (\bar{\underset{\sim}{x}} = \underset{\sim}{a} - i\underset{\sim}{b})$$

$$= \underset{\sim}{a}^T A \underset{\sim}{a} + i\,\underset{\sim}{a}^T A \underset{\sim}{b} - i\,\underset{\sim}{b}^T A \underset{\sim}{a} + \underset{\sim}{b}^T A \underset{\sim}{b}$$

$(n\times 1)^T \; n\times n \; n\times 1$

$$= (\underset{\sim}{b}^T A \underset{\sim}{a})^T = \underset{\sim}{a}^T A^T \underset{\sim}{b} = \underset{\sim}{a}^T A \underset{\sim}{b}$$

$$= \underset{\sim}{a}^T A \underset{\sim}{a} + \underset{\sim}{b}^T A \underset{\sim}{b}$$

Also

$$\lambda \bar{\underset{\sim}{x}}^T \underset{\sim}{x} = \lambda (\underset{\sim}{a} - i\underset{\sim}{b})^T (\underset{\sim}{a} + i\underset{\sim}{b})$$

$$= \lambda (\underset{\sim}{a}^T \underset{\sim}{a} + i\underset{\sim}{a}^T \underset{\sim}{b} - i\underset{\sim}{b}^T \underset{\sim}{a} + \underset{\sim}{b}^T \underset{\sim}{b})$$

$$= \lambda (\underset{\sim}{a}^T \underset{\sim}{a} + \underset{\sim}{b}^T \underset{\sim}{b})$$

$$\Rightarrow \underset{\sim}{a}^T A \underset{\sim}{a} + \underset{\sim}{b}^T A \underset{\sim}{b} = \lambda (\underset{\sim}{a}^T \underset{\sim}{a} + \underset{\sim}{b}^T \underset{\sim}{b})$$

$\iff$

$$\lambda = \frac{\underset{\sim}{a}^T A \underset{\sim}{a} + \underset{\sim}{b}^T A \underset{\sim}{b}}{\underset{\sim}{a}^T \underset{\sim}{a} + \underset{\sim}{b}^T \underset{\sim}{b}} \in \mathbb{R}.$$

So λ is real valued.

$\implies$ If $A\underset{\sim}{x} = \lambda \underset{\sim}{x}$ for $\underset{\sim}{x} \in \mathbb{C}^n$, then $\mathrm{Re}(\underset{\sim}{x})$ is an eigenvector corresponding to λ. & $\underset{\sim}{a}$

$$A(\underset{\sim}{a} + i\underset{\sim}{b}) = \lambda(\underset{\sim}{a} + i\underset{\sim}{b})$$

$$\iff A\underset{\sim}{a} + iA\underset{\sim}{b} = \lambda\underset{\sim}{a} + i\lambda\underset{\sim}{b}$$

$$\iff A\underset{\sim}{a} = \lambda\underset{\sim}{a}, \quad A\underset{\sim}{b} = \lambda\underset{\sim}{b}.$$

$z_1 = a_1 + ib_1$

$z_2 = a_2 + ib_2$

$z_1 = z_2 \iff a_1 = a_2$, $b_1 = b_2$

3. Eigenspaces corresponding to ~~different~~ distinct eigenvalues are orthogonal.

(3)

A symm. ($n \times n$). Let $\lambda_1 \neq \lambda_2$ be eigenvalues of A.

Suppose $\underset{\sim}{v}_1$ satisfies $A\underset{\sim}{v}_1 = \lambda_1 \underset{\sim}{v}_1$

and $\underset{\sim}{v}_2$ satisfies $A\underset{\sim}{v}_2 = \lambda_2 \underset{\sim}{v}_2$.

Then
$$\lambda_2 \underset{\sim}{v}_1^T \underset{\sim}{v}_2 = \underset{\sim}{v}_1^T A \underset{\sim}{v}_2$$
$$= \underset{\sim}{v}_1^T A^T \underset{\sim}{v}_2$$
$$= (A \underset{\sim}{v}_1)^T \underset{\sim}{v}_2$$
$$= \lambda_1 \underset{\sim}{v}_1^T \underset{\sim}{v}_2$$

$$\Leftrightarrow \underbrace{(\lambda_2 - \lambda_1)}_{\neq 0} \underset{\sim}{v}_1^T \underset{\sim}{v}_2 = 0$$

$$\Leftrightarrow \underset{\sim}{v}_1^T \underset{\sim}{v}_2 = 0.$$ □

Spectral decomposition

For a symmetric $n \times n$ matrix $\mathbf{A}$ with eigenvalues $\lambda_1, \dots, \lambda_n$ and corresponding unit-norm eigenvectors $\mathbf{u}_1, \dots, \mathbf{u}_n$, the representation

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^T = \lambda_1 \mathbf{u}_1 \mathbf{u}_1^T + \dots \lambda_n \mathbf{u}_n \mathbf{u}_n^T. = U \Lambda U^T$$

is called the *spectral decomposition* of $\mathbf{A}$. $U = [u_1, \dots, u_n]$

$$\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

Exercise: The matrix $\mathbf{A} = \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix}$ has unit eigenvectors

$$\mathbf{u}_1 = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} \quad \text{and} \quad \mathbf{u}_2 = \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$

corresponding to the eigenvalues $\lambda_1 = 8$ and $\lambda_2 = 3$, respectively.

Check that $\mathbf{A} = \lambda_1 \mathbf{u}_1 \mathbf{u}_1^T + \lambda_2 \mathbf{u}_2 \mathbf{u}_2^T$.

$$8\begin{pmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}\begin{pmatrix} 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix} + 3\begin{pmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}\begin{pmatrix} -1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix}$$

$$= 8\begin{pmatrix} 4/5 & 2/5 \\ 2/5 & 1/5 \end{pmatrix} + 3\begin{pmatrix} 1/5 & -2/5 \\ -2/5 & 4/5 \end{pmatrix}$$

$$= \begin{pmatrix} 35/5 & \frac{10}{5} \\ 10/5 & \frac{20}{5} \end{pmatrix} = \begin{pmatrix} 7 & 2 \\ 2 & 4 \end{pmatrix}$$

Result (Rank of a symmetric matrix)

For a symmetric matrix, the rank is equal to the number of nonzero eigenvalues.

In the "number" we count multiplicities.

Discuss how the Spectral Theorem implies the above.

$A_{n \times n}$ rank A = dim Col A

dim Col A + dim Nul A = n

dim Col A = sum of multiplicities of nonzero eigenvalues.

dim Nul A = mult. of $\lambda = 0$ as a root of ch. polynomial.

$\text{Nul } A = \{ \underset{\sim}{x} : A\underset{\sim}{x} = \underset{\sim}{0} \}$.

If $\text{Nul } A \neq \{\underset{\sim}{0}\}$ then $\lambda = 0$ is an eigenvalue of A. ($A\underset{\sim}{x} = 0\,\underset{\sim}{x}$ for nonzero $\underset{\sim}{x}$)

by ② from spectral thm.

So $\text{Nul } A$ = eigenspace corresponding to $\lambda = 0$.

↑

dimension is given by multiplicity of 0 as a root of the ch. poly.

Quadratic form

A *quadratic form* on $\mathbb{R}^n$ is a function on $\mathbb{R}^n$ given by $Q(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x}$ for some symmetric matrix $\mathbf{A}$. The matrix $\mathbf{A}$ is called the *matrix of the quadratic form*.

symm.

Classifications of quadratic forms

setminus

A quadratic form $Q(\mathbf{x})$ is

$\forall \mathbf{x} \in \mathbb{R}^n \setminus \{0\}$

1. *positive definite* if $Q(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{0}$.
2. *negative definite* if $Q(\mathbf{x}) < 0$ for all $\mathbf{x} \neq \mathbf{0}$.
3. *positive semidefinite* if $Q(\mathbf{x}) \geq 0$ for all $\mathbf{x}$.
4. *negative semidefinite* if $Q(\mathbf{x}) \leq 0$ for all $\mathbf{x}$.
5. *indefinite* if $Q(\mathbf{x})$ takes both positive and negative values.

We apply the same terms to the matrices of quadratic forms: E.g. a *positive definite matrix* $\mathbf{A}$ is a symmetric matrix such that $\mathbf{x}^T\mathbf{A}\mathbf{x}$ is positive definite.

Result (Symmetry vis-à-vis quadratic forms)

Let $\mathbf{A}$ *be an* $n \times n$ *matrix. We have*

1. $\mathbf{x}^T\mathbf{A}\mathbf{x} = \mathbf{x}^T\tilde{\mathbf{A}}\mathbf{x}$ *for all* $\mathbf{x} \in \mathbb{R}^n$*, where* $\tilde{\mathbf{A}} = (1/2)(\mathbf{A} + \mathbf{A}^T)$.
2. *If* $\mathbf{A}$ *is symmetric, then* $\mathbf{x}^T\mathbf{A}\mathbf{x} = 0$ *for all* $\mathbf{x} \in \mathbb{R}^n$ *implies* $\mathbf{A} = \mathbf{0}$.
3. $\mathbf{x}^T\mathbf{A}\mathbf{x} = 0$ *for all* $\mathbf{x} \in \mathbb{R}^n$ *implies* $\mathbf{A} = -\mathbf{A}^T$.

Prove the above results.

(2) $\underset{n\times n}{A}$

$$\underset{\sim}{x}^T A \underset{\sim}{x} = [x_1 \cdots x_n]\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & & a_{nn}\end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_n\end{bmatrix}$$

not symmetric

bonus

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} x_i a_{ij} x_j$$

$$x^T \tilde{A} x = x^T \left(\frac{1}{2} (A + A^T) \right) x$$

$\tilde{A} = \frac{1}{2}(A + A^T)$

$\tilde{A}^T = \frac{1}{2}(A + A^T)^T$

$= \frac{1}{2}(A^T + A)$

$= \tilde{A}$

$$= \frac{1}{2} x^T A x + \frac{1}{2} \underbrace{x^T A^T x}_{\text{scalar}}$$

$$= \frac{1}{2} x^T A x + \frac{1}{2} (x^T A^T x)^T$$

$$= \frac{1}{2} x^T A x + \frac{1}{2} x^T A x$$

$$= x^T A x$$

❷ *If* $\mathbf{A}$ *is symmetric, then* $\mathbf{x}^T\mathbf{A}\mathbf{x} = 0$ *for all* $\mathbf{x} \in \mathbb{R}^n$ *implies* $\mathbf{A} = \mathbf{0}$.

$\underset{n \times n}{A}$ symm.

$x^T A x = 0 \quad \forall x \in \mathbb{R}^n$ then...

$e_j = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \end{bmatrix} \leftarrow$ position j

$$\begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$x = e_j \Rightarrow \quad x^T A x = e_j^T A e_j$

$= a_{jj}$

$= 0 \quad \forall j.$

$$= \begin{bmatrix} a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= a_{22}$$

$x = e_i + e_j, \quad i \neq j$

$$x^T A x = (e_i + e_j)^T A (e_i + e_j)$$

$$= \underbrace{e_i^T A e_i}_{=0} + e_i^T A e_j + e_j^T A e_i + \underbrace{e_j^T A e_j}_{=0}$$

$$= a_{ij} + a_{ji}$$

symm.

$$= 2 a_{ij}$$

$$= 0 \quad \forall \quad i, j.$$

So $A = 0$.

3 $\mathbf{x}^T \mathbf{A} \mathbf{x} = 0$ *for all* $\mathbf{x} \in \mathbb{R}^n$ *implies* $\mathbf{A} = -\mathbf{A}^T$.

$A_{n \times n}$ not necessarily symm.

$$x^T A x = 0 \quad \forall \quad x \in \mathbb{R}^n$$

$$\Rightarrow \quad x^T A x + x^T A x = 0$$

$$\Rightarrow \quad x^T A x + (x^T A x)^T = 0$$

$$\Rightarrow \quad x^T A x + x^T A^T x = 0$$

$$\Rightarrow \quad x^T \underline{(A + A^T)} x = 0 \quad \forall \quad x \in \mathbb{R}^n$$

symm.

$\Rightarrow \quad A + A^T = 0 \quad$ by (2)

$\Leftrightarrow \quad A = -A^T.$

Theorem (Principal axes theorem)

Let $\mathbf{A}$ *be an* $n \times n$ *symmetric matrix. Then there is an orthogonal change of variable* $\mathbf{x} = \mathbf{P}\mathbf{y} \iff \mathbf{y} = \mathbf{P}^{-1}\mathbf{x}$ *which transforms the quadratic form* $\mathbf{x}^T\mathbf{A}\mathbf{x}$ *into a quadratic form* $\mathbf{y}^T\mathbf{D}\mathbf{y}$ *in which* $\mathbf{D}$ *is a diagonal matrix.*

We call the columns of $\mathbf{P}$ the *principal axes* of the quadratic form $\mathbf{x}^T\mathbf{A}\mathbf{x}$.

Prove the result.

Spectral decomp; diagonal

$$x^T A x = x^T P D P^{-1} x = y^T D y$$

y^T; $y = P^T x = P^{-1} x$ (because $P^T = P^{-1}$)

Theorem (Quadratic forms and eigenvalues)

Let $\mathbf{A}$ *be a symmetric* $n \times n$ *matrix. Then the quadratic form* $\mathbf{x}^T\mathbf{A}\mathbf{x}$ *is*

1. *positive definite if and only if the eigenvalues of* $\mathbf{A}$ *are all positive.*
2. *negative definite if and only if the eigenvalues of* $\mathbf{A}$ *are all negative.* ←
3. *indefinite if and only if* $\mathbf{A}$ *has both positive and negative eigenvalues.*

Prove the result. $\underset{n\times n}{A}$

$$P^T = P^{-1}$$

$$x^T A x > 0 \quad \forall\, x \in \mathbb{R}^n \setminus \{0\} \iff \underbrace{x^T P}_{y^T} D \underbrace{P^T x}_{y} > 0 \quad \forall\, x \in \mathbb{R}^n \setminus \{0\}$$

$$\iff y^T D y > 0 \quad \forall\, y \in \mathbb{R}^n \setminus \{0\}$$

$$\iff \quad \lambda_1 y_1^2 + \cdots + \lambda_n y_n^2 > 0 \quad \forall \, (y_1, \ldots, y_n) \in \mathbb{R}^n \setminus \{\underset{\sim}{0}\}$$

$$\iff \quad \lambda_1, \ldots, \lambda_n > 0 \, .$$

1. Spectral decomposition

$$\underset{n\times n,\ \text{symm}}{A} = P D P^{-1}, \quad P^{-1} = P^T$$

(D: Diagonal) — Spectral decomposition

2. Quadratic forms

3. Singular value decomposition

$$\underset{m\times n}{A} = U \Sigma V^T$$

$$\Sigma = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$$

Not all matrices admit a factorization like $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$, with $\mathbf{D}$ diagonal.

But any matrix of any dimension $m \times n$ has a factorization like

$$\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T$$

called its *singular value decomposition*.

$A_{m \times n}$

Singular values of an $m \times n$ matrix

The *singular values* of $\mathbf{A}$ are the square roots of the eigenvalues of $\mathbf{A}^T\mathbf{A}$.

Typically denote singular values by $\sigma_1 \geq \cdots \geq \sigma_n$.

Discuss: How do we know that the eigenvalues of $\mathbf{A}^T\mathbf{A}$ are nonnegative?

$\underset{\sim}{x}^T A^T A \underset{\sim}{x} = \|A\underset{\sim}{x}\|^2 \geq 0 \implies$ eigenvalues of A^TA are all ≥ 0.

Result (Setup for singular value decomposition)

Let $\mathbf{v}_1, \dots, \mathbf{v}_n$ *be an orthogonal basis for* $\mathbb{R}^n$ *of unit eigenvectors of* $\mathbf{A}^T\mathbf{A}$ *arranged so that the corresponding eigenvalues of* $\mathbf{A}^T\mathbf{A}$ *satisfy* $\lambda_1 \geq \cdots \geq \lambda_n$*. Then:*

1. *The singular values of* $\mathbf{A}$ *are the lengths of* $\mathbf{A}\mathbf{v}_1, \dots, \mathbf{A}\mathbf{v}_n$*.*
2. *If* $\mathbf{A}$ *has* r *nonzero singular values, then* $\{\mathbf{A}\mathbf{v}_1, \dots, \mathbf{A}\mathbf{v}_r\}$ *is an orthogonal basis for* $\text{Col}\,\mathbf{A}$ *and* $\text{rank}\,\mathbf{A} = r$*.*

Prove the results.

$\text{col } A^T A = \text{col } A^T = \text{row } A$

rank A = rank A^TA

A^TA has r nonzero eigenvalues.

Theorem (Singular value decomposition)

For $\underset{m\times n}{\mathbf{A}}$ *with rank r there exist orthogonal matrices* $\underset{m\times m}{\mathbf{U}}$ *and* $\underset{n\times n}{\mathbf{V}}$ *such that*

r = dim Col A = dim Row A

r ≤ min{m, n}

$$\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T, \quad \text{where} \quad \underset{m\times n}{\boldsymbol{\Sigma}} = \begin{bmatrix} \underset{r\times r}{\mathbf{D}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix},$$

m×n m×m n×n m×n m×n

where $\underset{r\times r}{\mathbf{D}}$ *is diag. with diag. entries* $\sigma_1 \geq \cdots \geq \sigma_r$ *the nonzero singular vals of* $\mathbf{A}$.

Note: The matrices $\mathbf{U}$ and $\mathbf{V}$ are not uniquely determined by $\mathbf{A}$, but $\boldsymbol{\Sigma}$ is.

The representation of $\mathbf{A}$ as $\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T$ is the *singular value decomposition* of $\mathbf{A}$.

Columns in $\mathbf{U}$ and $\mathbf{V}$ are, respectively, *left* and *right singular vectors* of $\mathbf{A}$.

orthogonal set of unit eigenvectors of A^TA ($n \times n$)

Prove SVD theorem: Show that we have $\mathbf{AV} = \mathbf{U\Sigma}$ under the choices

$$\underset{n\times n}{\mathbf{V}} = [\mathbf{v}_1 \dots \mathbf{v}_n], \quad \text{and} \quad \underset{m\times m}{\mathbf{U}} = [\sigma_1^{-1}\mathbf{A}\mathbf{v}_1 \dots \sigma_r^{-1}\mathbf{A}\mathbf{v}_r, \mathbf{u}_{r+1} \dots \mathbf{u}_m],$$

where $\mathbf{u}_{r+1} \dots \mathbf{u}_m$ are any vectors completing an orthonormal basis for $\mathbb{R}^m$.

$$AV = A[v_1 \cdots v_n] = [Av_1 \cdots Av_n]$$

$$U\Sigma = [\sigma_1^{-1}Av_1 \cdots \sigma_r^{-1}Av_r, u_{r+1} \cdots u_m]\begin{bmatrix} \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix}$$

$$= [A\underset{\sim}{v}_1 \cdots A\underset{\sim}{v}_r]$$

$$AV = U\Sigma$$

$$A \underbrace{VV^T}_{I} = U\Sigma V^T$$

Exercise: Find a SVD for $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}$.

The Invertible Matrix Theorem (concluded)

Let A be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix.

u. $(\text{Col } A)^{\perp} = \{\mathbf{0}\}$.

v. $(\text{Nul } A)^{\perp} = \mathbb{R}^n$.

w. $\text{Row } A = \mathbb{R}^n$.

x. A has n nonzero singular values.

$$\Sigma = \begin{bmatrix} D_{r\times r} & 0 \\ 0 & 0 \end{bmatrix}$$

$$A = U \Sigma V^T = U_r D V_r^T$$

U_r: first r columns of U

Recipe for reduced singular value decomposition

For an $m \times n$ matrix $\mathbf{A}$ with rank r:

1. Obtain orthonormal set $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ of eigenvectors of $\mathbf{A}^T\mathbf{A}$ with corresponding eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$, of which r are nonzero.
2. Obtain nonzero singular values $\sigma_1 = \sqrt{\lambda_1}, \dots, \sigma_r = \sqrt{\lambda_r}$.
3. Set $\mathbf{U}_r = [\sigma_1^{-1}\mathbf{A}\mathbf{v}_1 \; \cdots \; \sigma_r^{-1}\mathbf{A}\mathbf{v}_r]$.
4. Set $\mathbf{V}_r = [\mathbf{v}_1 \; \cdots \; \mathbf{v}_r]$.
5. Set $\mathbf{D} = \operatorname{diag}(\sigma_1, \dots, \sigma_r)$.

$\sum_{j=1}^{r} \sigma_j u_j v_j^T$

Then the *reduced singular value decomposition* of $\mathbf{A}$ is $\mathbf{A} = \mathbf{U}_r\mathbf{D}\mathbf{V}_r^T$.

We can also construct a *low-rank approximation* to $\mathbf{A}$. For $1 \leq s < r$, set

$$\mathbf{A}_s = \mathbf{U}_s\mathbf{D}_s\mathbf{V}_s^T,$$

where $\mathbf{U}_s = [\mathbf{u}_1 \cdots \mathbf{u}_s]$, $\mathbf{D}_s = \operatorname{diag}(\sigma_1, \dots, \sigma_s)$, $\mathbf{V}_s = [\mathbf{v}_1 \; \cdots \; \mathbf{v}_s]$.

Useful in image compression!

```
# read in color png
library(png)
im_col <- readPNG(source = "your_image.png")

# convert to grayscale with formula from
# https://www.had2know.org/technology/rgb-to-gray-scale-converter.html
im <- 0.299*im_col[,,1] + 0.587*im_col[,,2] + 0.114 * im_col[,,3]
m <- dim(im)[1]
n <- dim(im)[2]

# obtain singular value decomposition
im_svd <- svd(im)
s <- 10 # choose number of singular vectors to keep
Us <- im_svd$u[,1:s]
Ds <- diag(im_svd$d[1:s])
Vs <- im_svd$v[,1:s]

# construct low-rank approximation
im_approx <- Us %*% Ds %*% t(Vs)

# keep grayscale values in [0,1]
im_approx[im_approx > 1] <- 1
im_approx[im_approx < 0] <- 0

# display image
asp <- m/n
plot(NA,ylim = c(0,asp),xlim = c(0,1),type = "n",
     xaxt = "n",yaxt = "n",bty = "n",xlab = NA,ylab = NA)
rasterImage(im_approx,0,0,asp,asp,interpolate=FALSE)
```

A $m \times n$

$A_s = U_s D_s V_s$


Original 2091 x 1667 image
26.6 Mb
Reduced SVD, rank = 5
0.1 Mb
Reduced SVD, rank = 10
0.3 Mb
Reduced SVD, rank = 20
0.6 Mb
Reduced SVD, rank = 40
1.2 Mb
Reduced SVD, rank = 80
2.3 Mb

Lay, D. C. (2003). *Linear algebra and its applications. Third edition.* Pearson Education.