SPECTRAL ANALYSIS OF TIME SERIES

The idea is to decompose the time series as a sum of sinusoidal functions of different frequencies.

Defn: Discrete Fourier Transform (DFT):

The DFT of a vector $(X_1,\dots,X_n)^T \in \mathbb{C}^n$ is the vector $D = (D_1,\dots,D_n)^T$ with entries given by

$$D_j = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} X_t \exp(\imath t \lambda_j), \quad j=1,\dots,n$$

where $\lambda_1,\dots,\lambda_n$ are the values (frequencies)

$$\left\{ \frac{k}{n}\cdot 2\pi, \quad k = -\left\lfloor \frac{n-1}{2} \right\rfloor,\dots,\left\lfloor \frac{n}{2} \right\rfloor \right\} \subset (-\pi,\pi].$$

What is the DFT? Define the $n\times n$ matrix E_n as

$$E_n = \left(\frac{1}{\sqrt{n}} \exp(-\imath t \lambda_j) \right)_{1\le t,j\le n} = \begin{bmatrix} \frac{1}{\sqrt{n}}\exp(-\imath 1 \lambda_1) & \cdots & \frac{1}{\sqrt{n}}\exp(-\imath 1 \lambda_n) \\ \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{n}}\exp(-\imath n \lambda_1) & \cdots & \frac{1}{\sqrt{n}}\exp(-\imath n \lambda_n) \end{bmatrix}.$$

For any $x \in \mathbb{C}$, we have $x = a + \imath b$ for some $a,b \in \mathbb{R}$.

Denote by $\bar{x}$ the complex conjugate $a - \imath b$ of x.

For a vector $\underset{\sim}{u} = (u_1,\dots,u_n)^T \in \mathbb{C}^n$, let $u^* = (\bar{u}_1,\dots,\bar{u}_n)$

For an $n\times m$ matrix $U \in \mathbb{C}^{n\times m}$ with columns $\underset{\sim}{u}_1,\dots,\underset{\sim}{u}_m$ let

$$U^* = \begin{bmatrix} \underset{\sim}{u}_1^* \\ \vdots \\ \underset{\sim}{u}_m^* \end{bmatrix}.$$

Now we may write

$$\underset{\sim}{D}_n = E_n^* \underset{\sim}{X}_n.$$

The columns $\underset{\sim}{E}_{n,1}, \ldots, \underset{\sim}{E}_{n,m}$ of E_n form an orthonormal basis for $\mathbb{C}^n$ under the inner product $\langle a, b \rangle = a^* b \quad \forall\, a, b \in \mathbb{C}^n$. Note that

$$\underset{\sim}{E}_{n,i}^* \underset{\sim}{E}_{n,j} = \frac{1}{n} \sum_{t=1}^{n} \exp(\iota t \lambda_i) \exp(-\iota t \lambda_j)$$

$$= \frac{1}{n} \sum_{t=1}^{n} \exp(\iota t (\lambda_i - \lambda_j))$$

$$= \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j, \end{cases}$$

so $E_n^* E_n = I_n$, where I_n is the $n \times n$ identity matrix.

Because of the above, we have

$$\underset{\sim}{D}_n = E_n^* \underset{\sim}{X}_n = (E_n^* E_n)^{-1} E_n^* \underset{\sim}{X}_n,$$

so that the entries of $\underset{\sim}{D}_n$ are the coefficients resulting from the regression of $X_1, \ldots, X_n$ onto the columns of E_n, i.e. from the projection of $\underset{\sim}{X}_n$ onto $\mathbb{C}^n$ (to which it already belongs) using the columns of E_n as a basis for $\mathbb{C}^n$.

Since $\underset{\sim}{X}_n \in \mathbb{C}^n = \mathrm{Col}(E_n)$, we have

$$\underset{\sim}{X}_n = E_n (E_n^* E_n)^{-1} E_n^* \underset{\sim}{X}_n,$$

giving

$$\underset{\sim}{X}_n = E_n \underset{\sim}{D}_n,$$

which can be written as

$$X_t = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} D_j \exp(-\iota t \lambda_j).$$

This is like an inverse to the DFT which gives the data $X_1, \ldots, X_n$ in terms of the coefficients $D_1, \ldots, D_n$.

Interpretation of DFT:

The relative magnitudes of $D_1, \ldots, D_n$ show the relative prevalence in $X_1, \ldots, X_n$ of the frequencies $\lambda_1, \ldots, \lambda_n$.

This brings us to the periodogram.

For $x \in \mathbb{C}$, let $|x| = \sqrt{x\bar{x}}$ (so for $x = a + bi$, $|x| = \sqrt{a^2 + b^2}$).

Defn: The periodogram of $X_1, \ldots, X_n$ is the function given by

$$I_n(\lambda) = \frac{1}{n}\left|\sum_{t=1}^{n} X_t \exp(-\imath t \lambda)\right|^2.$$

We have $I_n(\lambda_j) = |D_j|^2$ for $j = 1, \ldots, n$.

We can make a plot of the periodogram to see which frequencies are dominant in the data. If we want, we can use the inverse to the DFT to reconstruct the series using a smaller number of dominant frequencies.

* R example *

The "population" version of the periodogram is the spectral density function, which we discuss next.

Suppose $X_1, \ldots, X_n$ come from a stationary t.s. $\{X_t, t \in \mathbb{Z}\}$ with acvf $\gamma(\cdot)$ and consider taking the expection of the periodogram and finding its limit as $n \to \infty$. We have

$$\begin{aligned}
\mathbb{E}\, I_n(\lambda) &= \mathbb{E}\, \frac{1}{n}\left|\sum_{t=1}^{n} X_t \exp(-\imath t \lambda)\right|^2 \\
&= \frac{1}{n}\, \mathbb{E}\left(\sum_{t=1}^{n} X_t \exp(-\imath t \lambda)\right)\left(\sum_{s=1}^{n} X_s \exp(\imath s \lambda)\right) \\
&= \frac{1}{n} \sum_{t=1}^{n} \sum_{s=1}^{n} \exp(\imath \lambda (s-t))\, \mathbb{E} X_t X_s.
\end{aligned}$$

$$= \frac{1}{n} \sum_{t=1}^{n} \sum_{s=1}^{n} \exp\left(\imath\lambda(s-t)\right) \gamma(t-s)$$

$$= \sum_{h=-(n-1)}^{n-1} \left(1 - \frac{|h|}{n}\right) \exp\left(-\imath\lambda h\right) \gamma(h).$$

Taking the limit as $n \to \infty$ gives

$$\lim_{n\to\infty} \mathbb{E}\, I_n(\lambda) = \lim_{n\to\infty} \sum_{h=-(n-1)}^{n-1} \left(1 - \frac{|h|}{n}\right) \exp\left(-\imath\lambda h\right) \gamma(h)$$

$$= \lim_{n\to\infty} \sum_{h=-\infty}^{\infty} \left(1 - \frac{|h|}{n}\right) \mathbb{1}(|h|<n) \exp\left(-\imath\lambda h\right) \gamma(h)$$

$$= \sum_{h=-\infty}^{\infty} \exp\left(-\imath\lambda h\right) \gamma(h).$$

We now define the spectral density.

__Defn:__ The spectral density of a stationary t.s. $\{X_t, t \in \mathbb{Z}\}$ with acvf $\gamma(\cdot)$ is

$$f(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \exp\left(-\imath\lambda h\right) \gamma(h) \quad \text{for } -\infty < \lambda < \infty.$$

__Remark:__ So $f(\lambda) = \left(\frac{1}{2\pi}\right) \lim_{n\to\infty} \mathbb{E}\, I_n(\lambda).$

We now establish some properties of the spectral density function.

__Result:__ The spectral density has the following properties.

(a) $f(\lambda) = f(-\lambda)$

(b) $f(\lambda) \geq 0$ for all $\lambda \in (-\pi, \pi]$

(c) $\gamma(k) = \int_{-\pi}^{\pi} e^{\imath k\lambda} f(\lambda)\, d\lambda = \int_{-\pi}^{\pi} \cos(k\lambda) f(\lambda)\, d\lambda$

Proof: For (a), we have

$$f(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \left[\cos(h\lambda) + \imath \sin(h\lambda) \right] \gamma(h)$$

$$= \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \cos(h\lambda) \gamma(h) + \underbrace{\frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \sin(h\lambda) \gamma(h)}_{=0}$$

$$= \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \cos(-h\lambda) \gamma(h)$$

$$= f(-\lambda),$$

where the third equality comes from the fact that $\cos(\cdot)$ is an even function and that, since $\sin(\cdot)$ is an odd function,

$$\sum_{h=-\infty}^{\infty} \sin(h\lambda)\gamma(h) = \sum_{h=-\infty}^{-1} \sin(h\lambda)\gamma(h) + \underbrace{\sin(0\cdot\lambda)\gamma(0)}_{=0} + \sum_{h=1}^{\infty} \sin(h\lambda)\gamma(h)$$

$$= -\sum_{h=1}^{\infty} \sin(h\lambda)\gamma(h) + \sum_{h=1}^{\infty} \sin(h\lambda)\gamma(h)$$

$$= 0.$$

For (b), we note that since $f(\lambda) = \frac{1}{2\pi} \lim_{n\to\infty} \mathbb{E}\, I_n(\lambda)$, where $\mathbb{E}\, I_n(\lambda) \geq 0$ for each $n \geq 1$, as we previously showed, we have $f(\lambda) \geq 0$.

For (c), we have for any $k \in \mathbb{Z}$

$$\int_{-\pi}^{\pi} \exp(\imath k\lambda) f(\lambda)\, d\lambda = \int_{-\pi}^{\pi} \exp(\imath k\lambda) \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \exp(-\imath h\lambda) \gamma(h)\, d\lambda$$

$$= \int_{-\pi}^{\pi} \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \exp\left(\imath\lambda(k-h)\right) \gamma(h)\, d\lambda$$

$$= \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \gamma(h) \underbrace{\int_{-\pi}^{\pi} \exp\left(\imath\lambda(k-h)\right) d\lambda}_{= \begin{cases} 2\pi, & \text{if } k=h \\ 0, & \text{if } k\neq h \end{cases}}$$

$$= \gamma(k).$$

Example: Spectral density of $WN(0,\sigma^2)$. The autocovariance function of $\{Z_t, t\in\mathbb{Z}\} \sim WN(0,\sigma^2)$ is

$$\gamma(h) = \begin{cases} \sigma^2 & \text{for } h=0 \\ 0 & \text{for } h\neq 0, \end{cases}$$

so that the spectral density is

$$f(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \exp(-\imath h\lambda)\gamma(h) = \frac{\gamma(0)}{2\pi} = \frac{\sigma^2}{2\pi}$$

for all λ.

The following result gives an expression for the periodogram in terms of the sample autocovariance function.

Result: The periodogram based on $X_1, \ldots, X_n \in \mathbb{R}$ satisfies

$$I_n(\lambda_j) = \sum_{h=-(n-1)}^{n-1} \hat{\gamma}_n(h) e^{-\imath h \lambda_j} \quad \text{for } j=1,\ldots,n,$$

where $\lambda_1, \ldots, \lambda_n$ are the Fourier frequencies

$$\left\{ \frac{k}{n}\cdot 2\pi, \; k = \lfloor \tfrac{n-1}{2} \rfloor, \ldots, \lfloor \tfrac{n}{2} \rfloor \right\} \subset (-\pi, \pi]$$

and $\hat{\gamma}_n(\cdot)$ is the sample acvf.

Proof: We have

$$I_n(\lambda_j) = \frac{1}{n} \left| \sum_{t=1}^{n} X_t \exp(-\imath t \lambda_j) \right|^2$$

$$= \frac{1}{n}\left| \sum_{t=1}^{n} (X_t - \bar{X}_n)\exp(-\imath t\lambda_j) + \underbrace{\sum_{t=1}^{n} \bar{X}_n \exp(-\imath t \lambda_j)}_{=0} \right|^2$$

because λ_j a Fourier freq.

$$= \frac{1}{n} \sum_{t=1}^{n}\sum_{s=1}^{t} (X_t - \bar{X}_n)(X_s - \bar{X}_n)\exp(-\imath(t-s)\lambda_j)$$

$$= \sum_{h=-(n-1)}^{n-1} \frac{1}{n} \sum_{t=1}^{n-|h|} (X_t - \bar{X}_n)(X_{t+|h|} - \bar{X}_n)\exp(\imath h \lambda_j)$$

$$= \sum_{h=-(n-1)}^{n-1} \hat{\gamma}_n(h)\exp(\imath h\lambda_j).$$

This result tempts us to use $(2\pi)^{-1} I_n(\lambda)$ to estimate $f(\lambda)$. We find, however, that it is not a consistent estimator. We discuss this more later on.

SPECTRAL DENSITIES OF ARMA(p,q) PROCESSES

In this section we derive an expression for the spectral density of an ARMA(p,q) process in terms of the ARMA coefficients $\phi_1,\dots,\phi_p$, and $\theta_1,\dots,\theta_q$ and the white noise variance σ^2.

We begin with the following result:

__Result__ (Thm 4.4.1 B&D Theory). Let $\{Y_t, t\in\mathbb{Z}\}$ be a stationary, possibly complex-valued, t.s. with spectral density f_Y, and let

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j Y_{t-j} \quad \text{for all} \quad t\in\mathbb{Z}$$

for some $\{\psi_j, j\in\mathbb{Z}\}$ satisfying $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$. Then the spectral density of $\{X_t, t\in\mathbb{Z}\}$ is

$$f_X(\lambda) = \left|\psi(e^{-\imath j\lambda})\right|^2 f_Y(\lambda),$$

where $\psi(z) = \sum_{j=-\infty}^{\infty} \psi_j z^j.$

Proof: The acvf of $\{X_t, t \in \mathbb{Z}\}$ at lag h is

$$\mathbb{E} X_t \overline{X}_{t+h} = \mathbb{E} \sum_{j=-\infty}^{\infty} \psi_j Y_{t-j} \sum_{k=-\infty}^{\infty} \overline{\psi}_k \overline{Y}_{t+h-k}$$

$$= \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j \overline{\psi}_k \gamma(h-k+j)$$

$$= \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j \overline{\psi}_k \int_{-\pi}^{\pi} \exp(\imath(h-k+j)\lambda) f_Y(\lambda)\, d\lambda$$

$$= \int_{-\pi}^{\pi} \sum_{j=-\infty}^{\infty} \psi_j \exp(\imath j\lambda) \sum_{k=-\infty}^{\infty} \overline{\psi}_k \exp(-\imath k\lambda) \exp(\imath h\lambda) f_Y(\lambda)\, d\lambda$$

$$= \int_{-\pi}^{\pi} \exp(\imath h\lambda) \underbrace{\left| \sum_{j=-\infty}^{\infty} \psi_j \exp(\imath j\lambda) \right|^2 f_Y(\lambda)}_{f_X(\lambda)}\, d\lambda .$$

Since this is true for all $h \in \mathbb{Z}$, we see based on property (c) of the spectral density from pg. 4, that the spectral density of $\{X_t, t \in \mathbb{Z}\}$ is given by

$$f_X(\lambda) = \left| \sum_{j=-\infty}^{\infty} \psi_j \exp(\imath j\lambda) \right|^2 f_Y(\lambda),$$

which is the claim.

We can now quite easily derive the form of the spectral density for ARMA(p,q) processes.

Result: Let $\{X_t, t \in \mathbb{Z}\}$ be an ARMA(p,q) process defined by

$$\phi(B) X_t = \theta(B) Z_t \quad , \quad \{Z_t, t \in \mathbb{Z}\} \sim \mathrm{WN}(0,\sigma^2),$$

where $\phi(\cdot)$ and $\theta(\cdot)$ have no common zeroes and $\phi(\cdot)$ has no zeroes on the unit circle, and such that $\{X_t, t \in \mathbb{Z}\}$ is not necessarily causal or invertible.

Then the spectral density of $\{X_t, t \in \mathbb{Z}\}$ is given by

$$f_X(\lambda) = \frac{\sigma^2}{2\pi} \frac{|\theta(e^{-i\lambda})|^2}{|\phi(e^{-i\lambda})|^2}, \quad -\pi \le \lambda \le \pi.$$

__Proof:__ Whether $\{X_t, t \in \mathbb{Z}\}$ is causal or noncausal, we may write

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j}, \quad \text{with} \quad \sum_{j=-\infty}^{\infty} |\psi_j| < \infty.$$

Now, set $U_t = \phi(B) X_t = \theta(B) Z_t$, and note that by the previous result we may write

$$f_U(\lambda) = |\phi(e^{-i\lambda})|^2 f_X(\lambda) = |\theta(e^{-i\lambda})|^2 \frac{\sigma^2}{2\pi}, \quad -\pi \le \lambda \le \pi,$$

where $f_U(\lambda)$ is the spectral density of $\{U_t, t \in \mathbb{Z}\}$ and where $\sigma^2/(2\pi)$ is the $WN(0,\sigma^2)$ spectral density. Rearranging this gives

$$f_X(\lambda) = \frac{\sigma^2}{2\pi} \frac{|\theta(e^{-i\lambda})|^2}{|\phi(e^{-i\lambda})|^2}.$$

__Example__ (Spectral density of MA(1) process):

Let $X_t = Z_t + \theta Z_{t-1}$ for all $t \in \mathbb{Z}$, where $\{Z_t, t \in \mathbb{Z}\} \sim WN(0,\sigma^2)$.

Then the spectral density of $\{X_t, t \in \mathbb{Z}\}$ is

$$\begin{aligned} f_X(\lambda) &= \frac{\sigma^2}{2\pi} |1 + \theta e^{-i\lambda}|^2 \\ &= \frac{\sigma^2}{2\pi} \left[(1 + \theta\cos(\lambda))^2 + (\theta \sin(\lambda))^2 \right] \\ &= \frac{\sigma^2}{2\pi} \left[1 + 2\theta\cos(\lambda) + \theta^2\cos^2(\lambda) + \theta^2 \sin^2(\lambda) \right] \\ &= \frac{\sigma^2}{2\pi} \left[1 + 2\theta\cos(\lambda) + \theta^2 \right] \end{aligned}$$

for $-\pi < \lambda \le \pi$.

LAG-WINDOW ESTIMATOR OF THE SPECTRAL DENSITY

We have said that even though

$$\lim_{n\to\infty} E \frac{1}{2\pi} I_n(\lambda) = f(\lambda), \qquad -\pi \le \lambda \le \pi,$$

we should not use $(2\pi)^{-1} I_n(\lambda)$ as an estimator of $f(\lambda)$ because it is inconsistent, in spite of being asymptotically unbiased (the property expressed above).

The following result gives the asymptotic joint distribution of a vector of periodogram ordinates, and we can see from this result that $(2\pi)^{-1} I_n(\lambda)$ cannot be a consistent estimator of $f(\lambda)$.

<u>Result:</u> (From Thm 10.3.2 of B&D Theory):

Let $\{X_t, t\in\mathbb{Z}\}$ be a linear process

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j}, \qquad \{Z_t, t\in\mathbb{Z}\} \sim \text{IID}(0,\sigma^2),$$

where $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$, and denote by $f(\cdot)$ the spectral density of $\{X_t, t\in\mathbb{Z}\}$.

Let $I_n(\cdot)$ be the periodogram based on $X_1,\dots,X_n$. Then if $f(\lambda) > 0 \ \forall \ \lambda \in [-\pi,\pi]$, the periodogram ordinates

$$I_n(\lambda_1), \dots, I_n(\lambda_m), \quad \text{for any } \lambda_1,\dots,\lambda_m \in (0,\pi)$$

are asymptotically distributed as independent exponential random variables with means $(2\pi) f(\lambda_1), \dots, (2\pi) f(\lambda_m)$, respectively.

From this theorem, we have

$$\lim_{n\to\infty} \text{Var}\, I_n(\lambda) \to (2\pi)^2 f^2(\lambda) \quad \text{for any } \lambda \in (0,\pi).$$

We find, however, that we can construct a consistent estimator for $f(\lambda)$ by locally averaging or smoothing the periodogram ordinates at the Fourier frequencies. We get consistency because as $n\to\infty$, there are an increasing number of Fourier frequencies in the neighborhood $(\lambda-\delta, \lambda+\delta)$, $\delta>0$, of any $\lambda\in(0,\pi)$. See §10.4 of B&D Theory for details.

Instead of smoothing the periodogram, we will focus on another way to construct a consistent estimator of the spectral density. This type of estimator is called a lag-window estimator.

Recall that the periodogram is given by

$$I_n(\lambda) = \sum_{h=-(n-1)}^{n-1} \hat{\gamma}_n(h) \exp(-i\lambda h).$$

We consider now the fact that the value of the sample acvf $\hat{\gamma}_n(\cdot)$ at lag h is based on $n-|h|$ pairs of observed data, so that at greater lags, $\hat{\gamma}_n(\cdot)$ is based on fewer data points. As a result, the sample acvf has a larger variance at larger lags. The basic idea of lag-window estimation of the spectral density is to truncate the sample acvf such that we set its value to zero at larger lags. Then we compute the periodogram based on this truncated sample acvf.

The basic form of the lag-window estimator of the spectral density $f(\cdot)$ is

$$\hat{f}_L^{\text{basic}}(\lambda) = \frac{1}{2\pi} \sum_{|h| \le L} \hat{\gamma}_n(h) \exp(-i\lambda h),$$

so that the terms in the periodogram from $\hat{\gamma}_n(L+1), \ldots, \hat{\gamma}_n(n-1)$ are discarded. More generally, the lag-window estimator takes the following form:

Defn: Given a sample acvf $\hat{\gamma}_n(\cdot)$ and a choice of L, the lag-window estimator of the spectral density $f(\cdot)$ is given by

$$\hat{f}_L(\lambda) = \frac{1}{2\pi} \sum_{|h| \le L} w\left(\frac{h}{L}\right) \hat{\gamma}_n(h) \exp(-i\lambda h), \quad -\pi \le \lambda \le \pi,$$

where $w(\cdot)$ is an even, piecewise continuous function such that

(i) $w(x) = 0$ for all $|x| > 1$,

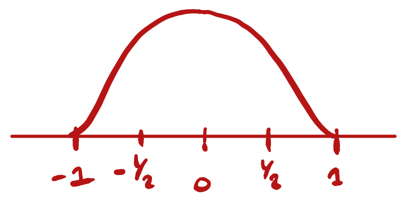
(ii) $w(0) = 1$,

(iii) $w(x) \le 1$ for all $|x| \le 1$.

Remark: The estimator $\hat{f}_L^{\text{basic}}(\lambda)$ used $w(x) = \mathbb{1}(|x| \le 1)$.

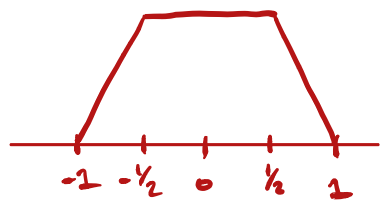
One example of a choice of the function $w(\cdot)$ is the Parzen window:

$$w(x) = \begin{cases} 1 - 6x^2 + 6|x|^3, & |x| < 1/2 \\ 2(1-|x|)^3, & 1/2 \le |x| \le 1 \\ 0, & |x| > 1. \end{cases}$$



Let's not forget the Trapezoid:

$$w(x) = \begin{cases} 1, & |x| < 1/2 \\ 1 - 2(|x| - 1/2), & 1/2 \le |x| \le 1 \\ 0, & |x| > 1 \end{cases}$$



Under an appropriate choice of L, the lag-window estimator of the spectral density is consistent, as the following result claims.

Result: Let $\{X_t, t \in \mathbb{Z}\}$ be the linear process

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j}, \quad \{Z_t, t \in \mathbb{Z}\} \sim \mathrm{IID}(0, \sigma^2)$$

with $\sum_{j=-\infty}^{\infty} |\psi_j| |j| < \infty$ and $\mathbb{E} Z_1^4 < \infty$. Then under choices of L such that $L \to \infty$ and $L/n \to 0$ as $n \to \infty$, we have

$$\hat{f}_L(\lambda) \to f(\lambda) \quad \text{in probability}$$

for each $\lambda \in [-\pi, \pi]$.

We conclude with a result which allows one to derive from a given spectral density the coefficients of the $MA(\infty)$ representation of the time series, provided the latter exists.

This result can be used to generate time series data from a process having a given spectral density.

Result: Let $\{X_t, t \in \mathbb{Z}\}$ be a stationary t.s. with spectral density f that satisfies

$$\int_{-\pi}^{\pi} \log f(\lambda)\, d\lambda > -\infty.$$

Then $\{X_t, t\in\mathbb{Z}\}$ has a unique $MA(\infty)$ representation

$$X_t = \sum_{j=0}^{\infty} c_j Z_{t-j}, \quad t \in \mathbb{Z},$$

where $\{Z_t, t\in\mathbb{Z}\} \sim WN(0,\sigma^2)$ and where the coefficients $\{c_j, j=0,1,2,\dots\}$ satisfy $\sum_{j=-\infty}^{\infty} |c_k|^2 < \infty$ and, moreover, can be found as follows:

Set

$$a_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp(-ik\lambda) \log f(\lambda)\, d\lambda, \qquad k=0,1,2,\dots$$

and $c_0 = 1$. Then $c_1, c_2, \dots$ are given by the recursion

$$c_{k+1} = \sum_{j=0}^{k} \left(1 - \frac{j}{k+1}\right) a_{k+1-j}\, c_j, \qquad k=0,1,2,\dots$$

In addition, $\sigma^2 = 2\pi \exp(a_0)$.

The above result is adapted from

Krampe, J., Kreiss, J. P., & Paparoditis, E. (2018). Estimated Wold representation and spectral-density-driven bootstrap for time series. JRSSB: Series B (Statistical Methodology), 80(4), 703-726.

To generate data from a t.s. with a given spectral density f, so long as $\int_{-\pi}^{\pi} \log f(\lambda) d\lambda > -\infty$, we can find the coefficients $c_0, c_1, c_2, \dots$ of the $MA(\infty)$ representation (truncating them at some large value) and then generate data from the moving average model.