MULTIVARIATE TIME SERIES

We now consider t.s. $\{\underset{\sim}{X}_t, t \in \mathbb{Z}\}$, where $\underset{\sim}{X}_t \in \mathbb{R}^m$, $t \in \mathbb{Z}$, so at each time point we observe m random variables.

We will assume throughout that $\mathbb{E} X_{tj}^2 < \infty$ for all $j = 1, \ldots, m$, $t \in \mathbb{Z}$.

Define the $m \times 1$ mean vectors and the $m \times m$ covariance matrices

$$\underset{\sim}{\mu}_t = \mathbb{E} \underset{\sim}{X}_t$$

$$\Gamma(t+h, t) = \mathbb{E}(\underset{\sim}{X}_{t+h} - \underset{\sim}{\mu}_{t+h})(\underset{\sim}{X}_t - \underset{\sim}{\mu}_t)^T, \quad h \in \mathbb{Z},$$

for $t \in \mathbb{Z}$. We can define stationarity in terms of these.

<u>**Defn**</u> (Stationarity for multivariate time series (m.t.s.)):

An m-dimensional m.t.s. $\{\underset{\sim}{X}_t, t \in \mathbb{Z}\}$ is stationary if

(i) $\mathbb{E} X_{tj}^2 < \infty$ for all $j = 1, \ldots, m$, $t \in \mathbb{Z}$

(ii) $\mathbb{E} \underset{\sim}{X}_t$ does not depend on t

(iii) For all $h \in \mathbb{Z}$, $\mathrm{Cov}(\underset{\sim}{X}_{t+h}, \underset{\sim}{X}_t)$ does not depend on t

If the above hold set $\underset{\sim}{\mu} = \mathbb{E} \underset{\sim}{X}_t$ and

$$\Gamma(h) = \mathrm{Cov}(\underset{\sim}{X}_{t+h}, \underset{\sim}{X}_t) = \mathbb{E} \underset{\sim}{X}_{t+h} \underset{\sim}{X}_t^T - \underset{\sim}{\mu}\underset{\sim}{\mu}^T \quad \text{for } h \in \mathbb{Z}.$$

The function $\Gamma(h)$ is called the covariance matrix function (cvmf).

Defining for $i, j = 1, \ldots, m$ the functions $\gamma_{ij}(\cdot)$ by

$$\gamma_{ij}(h) = \mathrm{Cov}(X_{t+h,i}, X_{t,j}),$$

we may write $\Gamma(h) = (\gamma_{ij}(h))_{1 \le i,j \le m}$.

The functions $\gamma_{11}(\cdot), \ldots, \gamma_{mm}(\cdot)$ are the acvfs of the component series of $\{\underset{\sim}{X}_t, t \in \mathbb{Z}\}$.

For each $i \neq j$, the function $\gamma_{ij}(\cdot)$ is called the cross-covariance function of the series $\{X_{ti}, t \in \mathbb{Z}\}$ and $\{X_{tj}, t \in \mathbb{Z}\}$.

It is important to note that $\gamma_{ij}(\cdot)$ is different from $\gamma_{ji}(\cdot)$:

We interpret $\gamma_{ij}(h) = Cov(X_{t+h,i}, X_{tj}) = Cov(X_{t,i}, X_{t-h,j})$ as the covariance between the current value of component i and the value of component j at h steps in the past.

Likewise $\gamma_{ji}(h) = Cov(X_{t+h,j}, X_{t,i})$ is the covariance between the current value of component i and the value of component j at h steps in the future.

We define the correlation matrix function (cmf) as

$$R(h) = \left(\rho_{ij}(h)\right)_{1 \leq i,j \leq m}, \quad h \in \mathbb{Z},$$

where $$\rho_{ij}(h) = \frac{\gamma_{ij}(h)}{\left[\gamma_{ii}(0)\,\gamma_{jj}(0)\right]^{1/2}}, \quad h \in \mathbb{Z}, \quad \text{for } i,j = 1, \ldots, m.$$

<u>Result</u> (Properties of cvmf of a stationary m.t.s.):

If $\Gamma(\cdot) = (\gamma_{ij}(\cdot))_{1 \leq i,j \leq m}$ is the cvmf of a stationary m.t.s. $\{\underset{\sim}{X}_t, t \in \mathbb{Z}\}$, then

(i) $\Gamma(h) = \Gamma^T(-h)$, for all $h \in \mathbb{Z}$

(ii) $|\gamma_{ij}(h)| \leq \left[\gamma_{ii}(0)\,\gamma_{jj}(0)\right]^{1/2}$, $i,j = 1, \ldots, m$, for all $h \in \mathbb{Z}$

(iii) $\gamma_{ii}(\cdot)$ is an acvf for $i = 1, \ldots, m$

(iv) For $n = 1, 2, \ldots$

$$\sum_{j=1}^{n} \sum_{k=1}^{n} \underset{\sim}{a}_j^T\, \Gamma(j-k)\, \underset{\sim}{a}_k \geq 0 \quad \text{for all } \underset{\sim}{a}_1, \ldots, \underset{\sim}{a}_n \in \mathbb{R}^m.$$

<u>Proof</u>: For (i), write

$$\Gamma(h) = \mathbb{E}\, \underset{\sim}{X}_{t+h} \underset{\sim}{X}_t^T - \underset{\sim}{\mu}\, \underset{\sim}{\mu}^T$$

$$= \mathbb{E}\, \underset{\sim}{X}_{(t-h)+t}\, \underset{\sim}{X}_{t-h}^{T} - \underset{\sim}{\mu}\underset{\sim}{\mu}^{T}$$

$$= \mathbb{E}\, \underset{\sim}{X}_{t}\, \underset{\sim}{X}_{t-h}^{T} - \underset{\sim}{\mu}\underset{\sim}{\mu}^{T}$$

$$= \left(\mathbb{E}\, \underset{\sim}{X}_{t-h}\, \underset{\sim}{X}_{t}^{T} - \underset{\sim}{\mu}\underset{\sim}{\mu}^{T}\right)^{T}$$

$$= \Gamma^{T}(-h).$$

For (ii), we have

$$|\gamma_{ij}(h)| = |Cov(X_{t+h,i}, X_{t,j})| \le [Var(X_{t+h,i})\, Var(X_{t,j})]^{1/2} = [\gamma_{ii}(0)\gamma_{jj}(0)]^{1/2}$$

Cauchy-Schwarz

For (iii), for each $i=1,\dots,m$, $\gamma_{ii}(h) = Cov(X_{t+h,i}, X_{t,i})$, so it is the acvf of $\{X_{ti}, t\in\mathbb{Z}\}$.

For (iv), for each $n=1,2,\dots$, for any $\underset{\sim}{a}_1,\dots,\underset{\sim}{a}_n \in \mathbb{R}^m$, we have

$$0 \le \mathbb{E}\left[\sum_{j=1}^{n} \underset{\sim}{a}_j^{T}(\underset{\sim}{X}_j - \underset{\sim}{\mu})\right]^2$$

$$= \mathbb{E} \sum_{j=1}^{n}\sum_{k=1}^{n} \underset{\sim}{a}_j^{T}(\underset{\sim}{X}_j - \underset{\sim}{\mu})(\underset{\sim}{X}_k - \underset{\sim}{\mu})^{T}\underset{\sim}{a}_k$$

$$= \sum_{j=1}^{n}\sum_{k=1}^{n} \underset{\sim}{a}_j^{T}\,\Gamma(j-k)\,\underset{\sim}{a}_k.$$

The cmf $R(\cdot)$ has properties (i)-(iv) above as well as the property

$$R_{ii}(0) = \rho_{ii}(0) = 1 \quad \text{for all} \quad i=1,\dots,m.$$

<u>Example</u>: Consider the 2-dimensional t.s. $\{\underset{\sim}{X}_t, t\in\mathbb{Z}\}$ with component series given by

$$X_{t1} = Z_t + \theta Z_{t-1} \quad \text{and} \quad X_{t2} = U_t + \xi Z_{t-1},$$

where $\{Z_t, t \in \mathbb{Z}\} \sim WN(0, \sigma_Z^2)$ and $\{U_t, t \in \mathbb{Z}\} \sim WN(0, \sigma_U^2)$ are white noise sequences which are uncorrelated with each other.

Then we have

h	$\gamma_{11}(h)$	$\gamma_{12}(h)$	$\gamma_{21}(h)$	$\gamma_{22}(h)$
2	0	0	0	0
1	$\theta\sigma_Z^2$	0	$\xi\sigma_Z^2$	0
0	$(1+\theta^2)\sigma_Z^2$	$\theta\xi\sigma_Z^2$	$\theta\xi\sigma_Z^2$	$\sigma_U^2 + \xi^2\sigma_Z^2$
-1	$\theta\sigma_Z^2$	$\xi\sigma_Z^2$	0	0
-2	0	0	0	0 ,

where we find the cross-covariances as

$$\gamma_{12}(0) = Cov(X_{t1}, X_{t2}) = Cov(Z_t + \theta Z_{t-1}, U_t + \xi Z_{t-1}) = \theta\xi\sigma_Z^2$$

$$\gamma_{12}(1) = Cov(X_{t+1,1}, X_{t2}) = Cov(Z_{t+1} + \theta Z_t, U_t + \xi Z_{t-1}) = 0$$

$$\gamma_{12}(-1) = Cov(X_{t-1,1}, X_{t2}) = Cov(Z_{t-1} + \theta Z_{t-2}, U_t + \xi Z_{t-1}) = \xi\sigma_Z^2$$

$$\gamma_{12}(h) = 0 \quad \text{for } |h| \geq 2$$

$$\gamma_{21}(0) = Cov(X_{t2}, X_{t1}) = \theta\xi\sigma_Z^2$$

$$\gamma_{21}(1) = Cov(X_{t+1,2}, X_{t1}) = Cov(U_{t+1} + \xi Z_t, Z_t + \theta Z_{t-1}) = \xi\sigma_Z^2$$

$$\gamma_{21}(-1) = Cov(X_{t-1,2}, X_{t1}) = Cov(U_{t-1} + \xi Z_{t-2}, Z_t + \theta Z_{t-1}) = 0$$

$$\gamma_{21}(h) = 0 \quad \text{for } |h| \geq 2.$$

Note that since $\Gamma(h) = \Gamma^T(-h)$, we have $\gamma_{12}(h) = \gamma_{21}(-h)$.

Then we have

$$\Gamma(0) = \begin{bmatrix} (1+\theta^2)\sigma_Z^2 & \theta\xi\sigma_Z^2 \\ \theta\xi\sigma_Z^2 & \sigma_U^2 + \xi^2\sigma_Z^2 \end{bmatrix}, \quad \Gamma(1) = \begin{bmatrix} \theta\sigma_Z^2 & 0 \\ \xi\sigma_Z^2 & 0 \end{bmatrix}, \quad \Gamma(-1) = \begin{bmatrix} \theta\sigma_Z^2 & \xi\sigma_Z^2 \\ 0 & 0 \end{bmatrix} = \Gamma^T(1)$$

We now define multivariate white noise.

<u>Defn</u> (Multivariate white noise):

An m-variate t.s. $\{\underset{\sim}{Z}_t, t \in \mathbb{Z}\}$ is called white noise with mean zero and covariance matrix Σ if it is stationary with mean vector $\underset{\sim}{0}$ and cvmf

$$\Gamma(h) = \begin{cases} \Sigma, & \text{if } h = 0 \\ 0, & \text{if } h \neq 0. \end{cases}$$

We write $\{\underset{\sim}{Z}_t, t \in \mathbb{Z}\} \sim WN(\underset{\sim}{0}, \Sigma)$.

If $\{\underset{\sim}{Z}_t, t \in \mathbb{Z}\}$ are independent identically distributed random vectors with mean $\underset{\sim}{0}$ and covariance matrix Σ, we write $\{\underset{\sim}{Z}_t, t \in \mathbb{Z}\} \sim IID(\underset{\sim}{0}, \Sigma)$.

Multivariate white noise is a building block for many multivariate time series models.

MULTIVARIATE LINEAR PROCESS

Just as in the univariate setting, we define in the multivariate setting a class of time series called linear processes. This is a m.t.s. which admits the representation

$$\underset{\sim}{X}_t = \sum_{j=-\infty}^{\infty} C_j \underset{\sim}{Z}_{t-j}, \quad t \in \mathbb{Z},$$

where $\{C_j, j \in \mathbb{Z}\}$ are matrices such that the components are absolutely summable and $\{\underset{\sim}{Z}_t, t \in \mathbb{Z}\} \sim WN(\underset{\sim}{0}, \Sigma)$.

Absolute summability of the components of $\{C_j, j \in \mathbb{Z}\}$ is the property that $\sum_{j=-\infty}^{\infty} |C_{j,k\ell}| < \infty$ for $k, \ell = 1, \ldots, m$, where $C_{j,k\ell}$ is the k,ℓ component of C_j.

A linear process is an $MA(\infty)$ process if $C_j = 0$ for $j < 0$, so that

$$\underset{\sim}{X}_t = \sum_{j=0}^{\infty} C_j \underset{\sim}{Z}_{t-j} \quad \text{for } t \in \mathbb{Z}.$$

<u>Exercise:</u> Show that the linear process is stationary and find its cvmf.

<u>Solution:</u> We have

$$\mathbb{E}\,\underset{\sim}{X}_t = \mathbb{E}\sum_{j=-\infty}^{\infty} C_j \underset{\sim}{Z}_{t-j} = \sum_{j=-\infty}^{\infty} C_j\, \mathbb{E}\,\underset{\sim}{Z}_{t-j} = 0.$$

by DCT, since C_j abs. summable.

In addition

$$\Gamma(h) = \operatorname{Cov}\left(\underset{\sim}{X}_{t+h}, \underset{\sim}{X}_t\right)$$

$$= \operatorname{Cov}\left(\sum_{j=-\infty}^{\infty} C_j \underset{\sim}{Z}_{t+h-j}, \sum_{k=-\infty}^{\infty} C_k \underset{\sim}{Z}_{t-k}\right)$$

$$= \sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} C_j \underbrace{\mathbb{E}\,\underset{\sim}{Z}_{t+h-j}\underset{\sim}{Z}_{t-k}^T}_{=\begin{cases}\Sigma, & j=k+h\\ 0, & \text{o.w.}\end{cases}} C_k^T$$

$$= \sum_{k=-\infty}^{\infty} C_{k+h}\, \Sigma\, C_k^T.$$

THE SAMPLE MEAN AND SAMPLE COVARIANCE FUNCTION

Given a stationary m.t.s. $\{\underset{\sim}{X}_t, t \in \mathbb{Z}\}$, we consider properties of the estimator

$$\bar{\underset{\sim}{X}}_n = \frac{1}{n}\sum_{t=1}^{n} \underset{\sim}{X}_t$$

for the mean $\underset{\sim}{\mu} = \mathbb{E}\,\underset{\sim}{X}_t$.

The next result gives conditions under which $\bar{\underset{\sim}{X}}_n$ is a consistent estimator of $\underset{\sim}{\mu}$.

__Result (Proposition 11.2.1 of B&D Theory):__

If $\{\underset{\sim}{X}_t, t \in \mathbb{Z}\}$ is a stationary m.t.s. with mean $\underset{\sim}{\mu}$ and cmf $\Gamma(\cdot)$, then

(i) $$\mathbb{E}\|\bar{\underset{\sim}{X}}_n - \underset{\sim}{\mu}\|_2^2 \to 0 \quad \text{as} \quad n \to \infty$$

if $\gamma_{ii}(h) \to 0$ for each $i = 1, \ldots, m$ as $h \to \infty$.

(ii) $$n\,\mathbb{E}\|\bar{\underset{\sim}{X}}_n - \underset{\sim}{\mu}\|_2^2 \longrightarrow \sum_{i=1}^{m} \sum_{h=-\infty}^{\infty} \gamma_{ii}(h)$$

if $\sum_{h=-\infty}^{\infty} |\gamma_{ii}(h)| < \infty$ for each $i = 1, \ldots, m$.

__Proof:__ We can see the above by noting

$$\mathbb{E}\|\bar{\underset{\sim}{X}}_n - \underset{\sim}{\mu}\|_2^2 = \mathbb{E}\sum_{i=1}^{m}(\bar{X}_{n,i} - \mu_i)^2 = \sum_{i=1}^{m} \mathrm{Var}(\bar{X}_{n,i})$$

and invoking previous results from the univariate setting.

The next result provides a condition under which the sample mean $\bar{\underset{\sim}{X}}_n$ is asymptotically Normal.

__Result (Prop 11.2.2 of B&D Theory):__

If $\{X_t, t \in \mathbb{Z}\}$ is an m-dimensional m.t.s. admitting the representation

$$X_t = \underset{\sim}{\mu} + \sum_{j=-\infty}^{\infty} C_j \underset{\sim}{Z}_{t-j}, \quad t \in \mathbb{Z},$$

where $\{\underset{\sim}{Z}_t, t \in \mathbb{Z}\} \sim \mathrm{IID}(0, \Sigma)$ and $\{C_j, j \in \mathbb{Z}\}$ satisfy $\sum_{j=-\infty}^{\infty} |C_{j,\ell k}| < \infty$ for each $\ell, k = 1, \ldots, m$. Then

$$\sqrt{n}(\bar{\underset{\sim}{X}}_n - \mu) \to N\left(\underset{\sim}{0}, \left(\sum_{j=-\infty}^{\infty} C_j\right) \Sigma \left(\sum_{j=-\infty}^{\infty} C_j\right)^T\right)$$

in distribution as $n \to \infty$.

The sample covariance matrix function is given by

$$\hat{\Gamma}_n(h) = \begin{cases} \frac{1}{n}\sum_{t=1}^{n-h}(\underset{\sim}{X}_{t+h} - \bar{\underset{\sim}{X}}_n)(\underset{\sim}{X}_t - \bar{X}_n)^T, & h = 0, \dots, n-1 \\ \frac{1}{n}\sum_{t=|h|+1}^{n}(\underset{\sim}{X}_{t-|h|} - \bar{X}_n)(\underset{\sim}{X}_t - \bar{X}_n)^T, & h = -(n-1), \dots, -1 \\ 0, & |h| = n, n+1, \dots \end{cases}$$

Recall that the sample autocovariance function for a univariate time series $\{X_t, t \in \mathbb{Z}\}$ is defined as

$$\hat{\gamma}_n(h) = \begin{cases} \frac{1}{n}\sum_{t=1}^{n-|h|}(X_{t+|h|} - \bar{X}_n)(X_t - \bar{X}_n), & h = -(n-1), \dots, 0, \dots, (n-1) \\ 0, & |h| = n, n+1, \dots \end{cases}$$

The reason $\hat{\Gamma}(h)$ is more complicated then $\hat{\gamma}_n(h)$ is that

$$\mathrm{Cov}(\underset{\sim}{X}_{t+h}, \underset{\sim}{X}_t) = \mathbb{E}\, \underset{\sim}{X}_{t+h}\underset{\sim}{X}_t^T - \underset{\sim}{\mu}\underset{\sim}{\mu}^T = (\mathbb{E}\, \underset{\sim}{X}_t \underset{\sim}{X}_{t+h} - \underset{\sim}{\mu}\underset{\sim}{\mu})^T = [\mathrm{Cov}(\underset{\sim}{X}_t, \underset{\sim}{X}_{t+h})]^T,$$

whereas for the univariate t.s. $\mathrm{Cov}(X_{t+h}, X_t) = \mathrm{Cov}(X_t, X_{t+h})$.

Denote by $\hat{\gamma}_{ij}(h)$, $i,j = 1, \dots, m$, the entries of $\hat{\Gamma}_n(h)$, for $h \in \mathbb{Z}$. Then the sample c.m.f. is defined by

$$\hat{R}_n(h) = (\hat{\rho}_{ij}(h))_{1 \le i,j \le m},$$

where

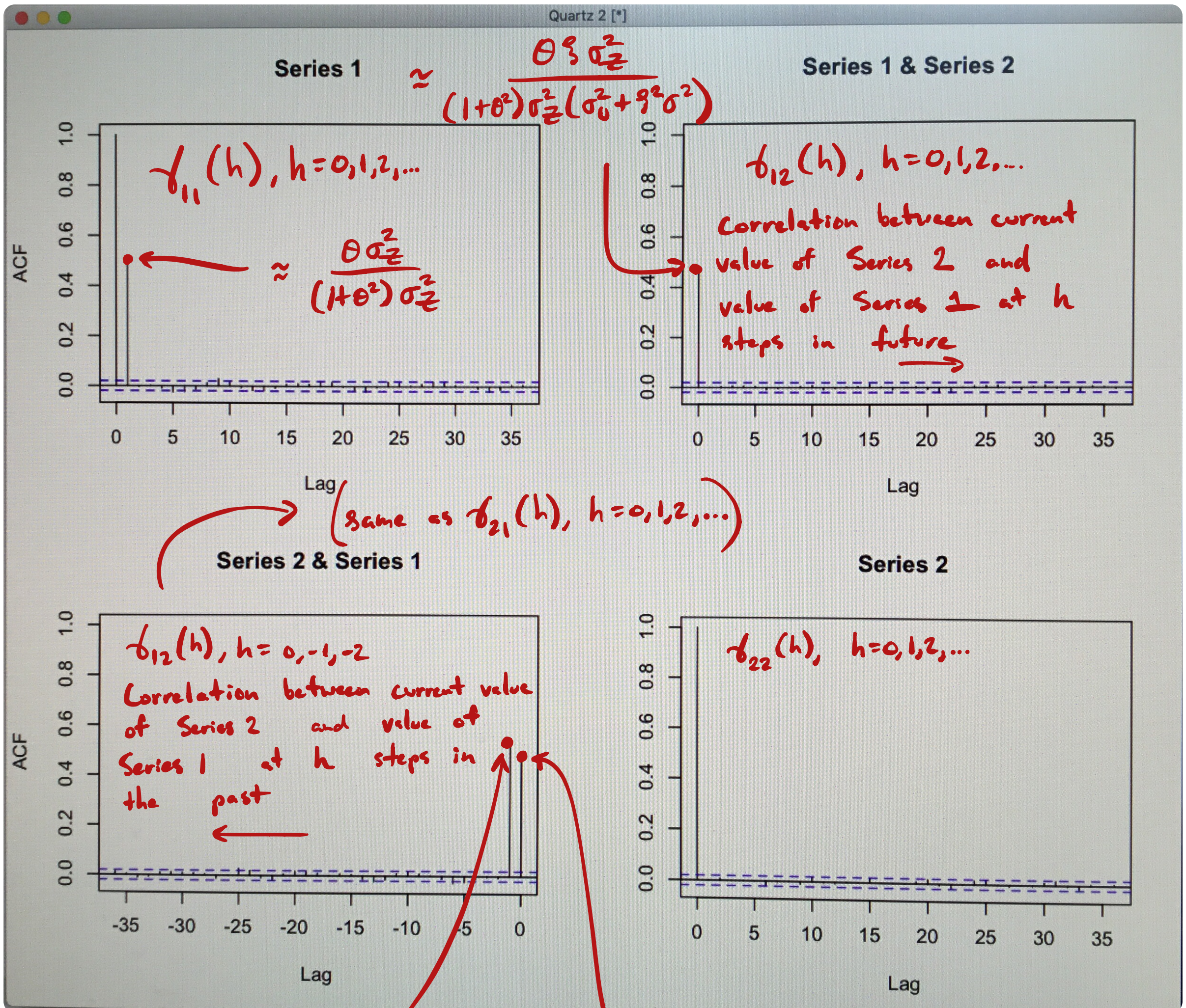
$$\hat{\rho}_{ij}(h) = \frac{\hat{\gamma}_{ij}(h)}{[\hat{\gamma}_{ii}(0)\hat{\gamma}_{jj}(0)]^{1/2}} \quad \text{for } i,j = 1, \dots, m, \; h \in \mathbb{Z}.$$

Note that $\hat{\gamma}_{11}(\cdot), \dots, \hat{\gamma}_{mm}(\cdot)$ and $\hat{\rho}_{11}(\cdot), \dots, \hat{\rho}_{mm}(\cdot)$ are the sample acvfs and acfs of the component time series.

Plotting the sample cmf

The sample c.m.f. based on a length 10,000 realization of this series from pg. 3 is plotted below in R under the settings

$$\theta = 0.9, \quad \xi = 0.5, \quad \sigma_Z^2 = 1, \quad \sigma_U^2 = 0.5$$



$$\approx \frac{\xi\sigma_Z^2}{(1+\theta^2)\sigma_Z^2(\sigma_U^2+\xi^2\sigma^2)} \qquad \approx \frac{\theta\xi\sigma_Z^2}{(1+\theta^2)\sigma_Z^2(\sigma_U^2+\xi^2\sigma^2)}$$

The R code acf(X, type="covariance"), where X is an $n \times m$ matrix containing in each column a time series, computes the sample cmf $\hat{\Gamma}_n(\cdot)$ and stores in an array of the form

$$\begin{bmatrix} \hat\gamma_{11}(0) & \hat\gamma_{21}(0) & \cdots & \hat\gamma_{m1}(0) \\ \vdots & \vdots & \ddots & \vdots \\ \hat\gamma_{11}(K) & \hat\gamma_{21}(K) & \cdots & \hat\gamma_{m1}(K) \end{bmatrix}$$

$$\vdots$$

$$\begin{bmatrix} \hat\gamma_{1m}(0) & \hat\gamma_{2m}(0) & \cdots & \hat\gamma_{mm}(0) \\ \vdots & \vdots & \ddots & \vdots \\ \hat\gamma_{1m}(K) & \hat\gamma_{2m}(K) & \cdots & \hat\gamma_{mm}(K) \end{bmatrix}, \quad \text{where } K \text{ is the maximum lag.}$$

MULTIVARIATE ARMA MODELS

Defn: A m.t.s. $\{\underset{\sim}{X}_t, t \in \mathbb{Z}\}$ is an m-variate ARMA(p,q) process if it is a stationary solution to

$$\underset{\sim}{X}_t - \Phi_1 \underset{\sim}{X}_{t-1} - \cdots - \Phi_p \underset{\sim}{X}_{t-p} = \underset{\sim}{Z}_t + \Theta_1 \underset{\sim}{Z}_{t-1} + \cdots + \Theta_q \underset{\sim}{Z}_{t-q}, \quad t \in \mathbb{Z},$$

where $\{\underset{\sim}{Z}_t, t \in \mathbb{Z}\} \sim WN(0, \Sigma)$ and $\Phi_1, \ldots, \Phi_p$ and $\Theta_1, \ldots, \Theta_q$ are $m \times m$ matrices.

Defining the $m \times m$-matrix-valued polynomials

$$\Phi(z) = I_m + \Phi_1 z + \cdots + \Phi_p z^p$$
$$\Theta(z) = \underbrace{I_m}_{m \times m \text{ identity matrix}} + \Theta_1 z + \cdots + \Theta_q z^q,$$

we may write the multivariate ARMA equations as

$$\Phi(B) \underset{\sim}{X}_t = \Theta(B) \underset{\sim}{Z}_t, \quad t \in \mathbb{Z}.$$

There are natural extensions of causality and invertibility to the case of multivariate ARMA models. See §11.3 of B&D Theory for details.

MULTIVARIATE SPECTRAL DENSITY

For m.t.s. the spectral density is a matrix-valued function.

Defn (Spectral density matrix function): If the cvmf $\Gamma(\cdot)$ satisfies

$$\sum_{k=-\infty}^{\infty} |\gamma_{ij}(k)| < \infty \quad \text{for all} \quad i,j = 1, \ldots, m,$$

then the spectral density matrix function is defined as

$$f(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \exp(-i\lambda h) \, \Gamma(h), \quad -\pi \le \lambda \le \pi.$$

Similarly to in the univariate setting, we have

$$\Gamma(h) = \int_{-\pi}^{\pi} f(\lambda) \exp(i\lambda h)\, d\lambda, \quad h \in \mathbb{Z}.$$

For the rest of this section, consider a stationary bivariate t.s. $\{\underset{\sim}{X}_t, t \in \mathbb{Z}\}$ with acvfs and cross-covariances $\gamma_{ij}(\cdot)$, $i,j=1,2$ satisfying

$$\sum_{h=-\infty}^{\infty} |\gamma_{ij}(h)| < \infty, \quad i,j=1,2.$$

Then the function

$$f_{12}(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \exp(-ih\lambda)\gamma_{12}(h), \quad -\pi \le \lambda \le \pi$$

is called the cross-spectrum or cross-spectral density of the time series $\{X_{t1}, t \in \mathbb{Z}\}$ and $\{X_{t2}, t \in \mathbb{Z}\}$.

We have

$$\begin{aligned} f_{21}(\lambda) &= \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \exp(-ih\lambda)\gamma_{21}(h) \\ &= \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \exp(ih\lambda)\gamma_{21}(-h) \\ &= \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \exp(ih\lambda)\gamma_{12}(h) \\ &= \overline{f_{12}(\lambda)} \end{aligned}$$

We see that the off-diagonals of the spectral density matrix are the cross-spectra.

$$f(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \exp(-ih\lambda)\Gamma(h) = \begin{bmatrix} f_{11}(\lambda) & f_{12}(\lambda) \\ f_{21}(\lambda) & f_{22}(\lambda) \end{bmatrix}.$$

Example: Consider again the m.t.s $\{\underset{\sim}{X}_t, t \in \mathbb{Z}\}$ defined by

$$X_{t1} = Z_t + \theta Z_{t-1}$$

$$X_{t2} = U_t + \phi Z_{t-1}, \qquad t \in \mathbb{Z},$$

where $\{Z_t, t \in \mathbb{Z}\} \sim WN(0, \sigma_Z^2)$ and $\{U_t, t \in \mathbb{Z}\} \sim WN(0, \sigma_U^2)$ are white noise sequences which are uncorrelated with each other.

Based on $\Gamma(h)$, $h \in \mathbb{Z}$ that we found before, we have

MA(1)

$$f_{11}(\lambda) = \frac{\sigma_Z^2}{2\pi} \left| 1 + \theta e^{i\lambda} \right|^2 = \cdots = \frac{\sigma_Z^2}{2\pi} \left[1 + 2\theta \cos(\lambda) + \theta^2 \right]$$

$$f_{22}(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \exp(-i\lambda h)\, \gamma_{22}(h) = \frac{\sigma_U^2 + \phi^2 \sigma_Z^2}{2\pi}$$

$$f_{12}(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \exp(-i\lambda h)\, \gamma_{12}(h)$$

$$= \frac{1}{2\pi} \left[\theta \phi \sigma_Z^2 + \exp(-i(-1)\lambda)\, \phi \sigma_Z^2 \right]$$

$$= \frac{1}{2\pi} \left[\theta \phi \sigma_Z^2 + \phi \sigma_Z^2 \left(\cos(\lambda) + i \sin(\lambda) \right) \right]$$

$$= \frac{1}{2\pi} \left[\phi \sigma_Z^2 (\theta + \cos(\lambda)) + i \phi \sigma_Z^2 \sin(\lambda) \right]$$

$$f_{21}(\lambda) = \frac{1}{2\pi} \left[\phi \sigma_Z^2 (\theta + \cos(\lambda)) - i \phi \sigma_Z^2 \sin(\lambda) \right]$$

So we have

$$f(\lambda) = \frac{\sigma_Z^2}{2\pi} \begin{bmatrix} 1 + 2\theta \cos(\lambda) + \theta^2 & \phi(\theta + \cos(\lambda)) + i \phi \sin(\lambda) \\ \phi(\theta + \cos(\lambda)) - i \phi \sin(\lambda) & \sigma_U^2 / \sigma_Z^2 + \phi^2 \end{bmatrix}$$

There are several functions related to the cross-spectrum between two time series which are of interest, in particular to the signal processing community. Consider the following:

Squared coherency function:

$$K_{12}^2(\lambda) = \frac{|f_{12}(\lambda)|^2}{f_{11}(\lambda)\, f_{22}(\lambda)}$$

This satisfies $0 \le K^2(\lambda) \le 1$ and is like the correlation between the time series at the frequency λ.

Co-spectrum: $c_{12}(\lambda) = \mathrm{Re}\{f_{12}(\lambda)\}$

Quadrature spectrum: $q_{12}(\lambda) = -\mathrm{Im}\{f_{12}(\lambda)\}$

Amplitude spectrum: $\alpha_{12}(\lambda) = |f_{12}(\lambda)| = \sqrt{c_{12}^2(\lambda) + q_{12}^2(\lambda)}$

Phase spectrum: $\phi_{12}(\lambda)$, where $\phi_{12}(\cdot)$ satisfies

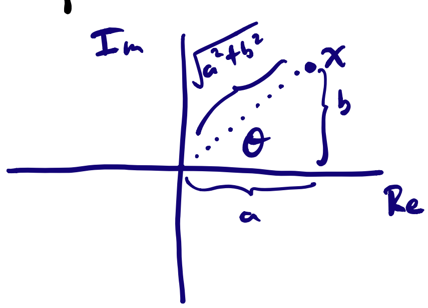
$$f_{12}(\lambda) = \alpha_{12} \exp\left[i\, \phi_{12}(\lambda)\right],$$

so $$\phi_{12}(\lambda) = \tan^{-1}\left(-\frac{q_{12}(\lambda)}{c_{12}(\lambda)}\right).$$

Recall that any complex number $x = a + bi$ can be expressed

$$x = \sqrt{a^2+b^2}\, \exp\left(i \tan^{-1}\left(\frac{b}{a}\right)\right),$$

according to this picture:



$$\theta = \tan^{-1}\left(\frac{b}{a}\right)$$

CHECKING FOR INDEPENDENCE OF TWO TIME SERIES

We now consider how to check whether two time series are independent of each other. Our first instinct is to look at the values of the sample cross-correlation function $\hat{\rho}_{12}(\cdot)$. This is right, but we find that the variance of $\hat{\rho}_{12}(h)$, $h=0,1,2,\ldots$ depends on the marginal correlation structures of the two time series, so the story is more complicated than checking whether $\hat{\rho}_{12}(\cdot)$ is "large" for some lags.

The following theorem gives insight.

__Result (from Thm 11.2.2 of B&D Theory):__

Consider two linear processes

$$X_{t1} = \sum_{j=-\infty}^{\infty} \alpha_j Z_{t-j} \quad , \quad \{Z_t\ t\in\mathbb{Z}\} \sim \text{IID}(0,\sigma_1^2)$$

$$X_{t2} = \sum_{j=-\infty}^{\infty} \beta_j U_{t-j} \quad , \quad \{U_t,\ t\in\mathbb{Z}\} \sim \text{IID}(0,\sigma_2^2),$$

where $\{Z_t, t\in\mathbb{Z}\}$ and $\{U_t, t\in\mathbb{Z}\}$ are independent, $\sum_{j=-\infty}^{\infty}|\alpha_j| < \infty$ and $\sum_{j=-\infty}^{\infty}|\beta_j| < \infty$. Then for each $h=0,1,2,\ldots$

$$\sqrt{n}\,\hat{\rho}_{12}(h) \rightarrow N\left(0, \sum_{j=-\infty}^{\infty} \rho_{11}(j)\rho_{22}(j)\right) \quad \text{in distribution}$$

as $n\to\infty$.

Note that the time series $\{X_{t1}, t\in\mathbb{Z}\}$ and $\{X_{t2}, t\in\mathbb{Z}\}$ defined in the result are independent. This result suggests plotting the values of $\hat{\rho}_{12}(h)$ for $h=0,1,2,\ldots K$, for some $K\geq 1$, and checking whether any fall outside of the bounds given by

$$\pm 1.96\sqrt{\sum_{j=-\infty}^{\infty} \rho_{11}(j)\rho_{22}(j)/n} \ .$$

This asymptotic variance is, however, not easily estimated. One way to proceed which obviates estimation of the limiting variance above is to "pre-whiten" one or both of the series.

Note that if one or the other or both of the time series $\{X_{t1}, t \in \mathbb{Z}\}$ and $\{X_{t2}, t \in \mathbb{Z}\}$ is/are white noise, then

$$\sum_{j=-\infty}^{\infty} \rho_{11}(j)\rho_{22}(j) = 1.$$

So to test whether a time series is correlated with a white noise sequence or to test whether two white noise sequences are correlated, we may check whether $\hat{\rho}_{12}(h)$ for $h = 0, 1, \ldots, K$, for some $K \geq 1$, falls outside of the bounds $\pm 1.96/\sqrt{n}$.

"Pre-whitening" can be done, for example, by obtaining the residuals of ARMA fits on both time series. Then $\hat{\rho}_{12}(\cdot)$ can be computed on the sets of residuals and then it can be compared to the bounds $\pm 1.96/\sqrt{n}$. If $\hat{\rho}_{12}(\cdot)$ falls far outside of these bounds for any lags, one will suspect that the original time series are not independent.

We now go through a proof of the result.

Proof of Result:

For any fixed value of $h = 0, 1, 2, \ldots$, we have

$$\hat{\gamma}_{12}(h) = \frac{1}{n}\sum_{t=1}^{n-h}\left(X_{t+h,1} - \bar{X}_1\right)\left(X_{t,2} - \bar{X}_2\right)$$

$$= \frac{1}{n}\sum_{t=1}^{n-h}\left[X_{t+h,1}X_{t,2} - X_{t+h,1}\bar{X}_2 - \bar{X}_1 X_{t,2} + \bar{X}_1\bar{X}_2\right]$$

$$= \frac{1}{n}\sum_{t=1}^{n} X_{t+h,1}X_{t,2} - \underbrace{\frac{1}{n}\sum_{t=n-h+1}^{n} X_{t+h,1}X_{t,2}}_{o_p(n^{-1/2})}$$

$$- \underbrace{\frac{1}{n}\sum_{t=1}^{n-h} X_{t+h,1}\bar{X}_2}_{o_p(n^{-1/2})} - \underbrace{\frac{1}{n}\sum_{t=1}^{n-h}\bar{X}_1 X_{t,2}}_{o_p(n^{-1/2})} + \underbrace{\left(\frac{n-h}{n}\right)\bar{X}_1\bar{X}_2}_{o_p(n^{-1/2})}$$

$$= \frac{1}{n}\sum_{t=1}^{n} X_{t+h,1}X_{t,2} + o_p(n^{-1/2}).$$

where $o_p(n^{-1/2})$ denotes a term which converges to zero even when multiplied by $n^{1/2}$.

Thus we have

$$\hat{\gamma}_{12}(h) = \gamma_{12}^{*}(h) + o_p(n^{-1/2})$$

where

$$\gamma_{12}^{*}(h) = \frac{1}{n}\sum_{t=1}^{n} X_{t+h,1} X_{t,2}.$$

The main part of the proof is establishing

$$(\star) \qquad \sqrt{n}\,\gamma_{12}^{*}(h) \longrightarrow N\left(0, \sum_{j=-\infty}^{\infty} \gamma_{11}(j)\gamma_{22}(j)\right) \text{ in dist}$$

as $n \to \infty$, from which the claim follows by simpler arguments.

To establish $(\star)$, we begin by defining for each $m = 1, 2, \ldots$ the random variable

$$\gamma_{12}^{*(m)}(h) = \frac{1}{n}\sum_{t=1}^{n} X_{t+h,1}^{(m)} X_{t,2}^{(m)},$$

where

$$X_{t1}^{(m)} = \sum_{|i|\le m} \alpha_i Z_{t-i} \quad \text{and} \quad X_{t2}^{(m)} = \sum_{|j|\le m} \beta_j U_{t-j} \quad \text{for} \quad t \in \mathbb{Z}.$$

Thus $\gamma_{12}^{*(m)}(h)$ is the mean of a length-n realization of the $(2m+1)$-dependent time series given by

$$Y_t^{(m)} = X_{t+h,1}^{(m)} X_{t,2}^{(m)}, \quad t \in \mathbb{Z}.$$

The time series $\{Y_t^{(m)}, t \in \mathbb{Z}\}$ has acvf given by

$$\begin{aligned}
\zeta^{(m)}(k) &= \mathbb{E}\, Y_{t+k}^{(m)} Y_t^{(m)} \\
&= \mathbb{E}\, X_{t+h+k,1}^{(m)} X_{t+k,2}^{(m)} X_{t+h,1}^{(m)} X_{t,2}^{(m)} \\
&= \mathbb{E} \sum_{|i|\le m} \alpha_i Z_{t+h+k-i} \sum_{|j|\le m} \beta_j U_{t+k-j} \sum_{|\ell|\le m} \alpha_\ell Z_{t+h-\ell} \sum_{|g|\le m} \beta_g U_{t-g}
\end{aligned}$$

$$= \sum_{|i|\le m} \sum_{|l|\le m} \alpha_i \alpha_l \, \mathbb{E}\, Z_{t+h+k-i} Z_{t+h-l} \sum_{|j|\le m} \sum_{|g|\le m} \beta_j \beta_g \, \mathbb{E}\, U_{t+k-j} U_{t-g}$$

$$= \sum_{l=-m}^{m-|k|} \alpha_{l+|k|} \alpha_l \, \sigma_1^2 \sum_{g=-m}^{m-|k|} \beta_{g+|k|} \beta_g \, \sigma_2^2 .$$

By results proven in Lecture 3 (CLT for m-dependent processes), we have

$$\sqrt{n}\, \gamma_{12}^{(m)}(h) \to N\left(0, \sum_{|k|\le m} \zeta^{(m)}(k)\right) \quad \text{in distribution}$$

as $n \to \infty$.

Now,

$$\lim_{m\to\infty} \sum_{|k|\le m} \zeta^{(m)}(k) = \lim_{m\to\infty} \sum_{|k|\le m} \sum_{l=-m}^{m-|k|} \alpha_{l+|k|} \alpha_l \, \sigma_1^2 \sum_{g=-m}^{m-|k|} \beta_{g+|k|} \beta_g \, \sigma_2^2$$

$$= \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \alpha_{l+|k|} \alpha_l \, \sigma_1^2 \sum_{g=-\infty}^{\infty} \beta_{g+|k|} \beta_g \, \sigma_2^2$$

$$= \sum_{k=-\infty}^{\infty} \gamma_{11}(k) \gamma_{22}(k),$$

using the fact that

$$\gamma_{11}(k) = \sum_{l=-\infty}^{\infty} \alpha_{l+|k|} \alpha_l \, \sigma_1^2 \quad \text{and} \quad \gamma_{22}(k) = \sum_{g=-\infty}^{\infty} \beta_{g+|k|} \beta_g \, \sigma_2^2 .$$

Thus, by the Useful Proposition of Lec 3 (Proposition 6.3.9 of B&D Theory), (*) holds if we can establish

$$\lim_{m\to\infty} \limsup_{n\to\infty} n \, \mathbb{E} \left| \gamma_{12}^{*}(h) - \gamma_{12}^{(m)}(h) \right| = 0.$$

We omit the details of showing this.

To invoke the Useful Proposition

- $\gamma_{12}^{*}(h)$ plays the role of X_n
- $\gamma_{12}^{(m)}(h)$ plays the role of W_{nj}
- A r.v. with dist. $N\left(0, \sum_{|k|\le m} \zeta(k)\right)$ plays the role of W_j
- A r.v. with dist. $N\left(0, \sum_{k=-\infty}^{\infty} \gamma_{11}(k)\gamma_{22}(k)\right)$ plays the role of W