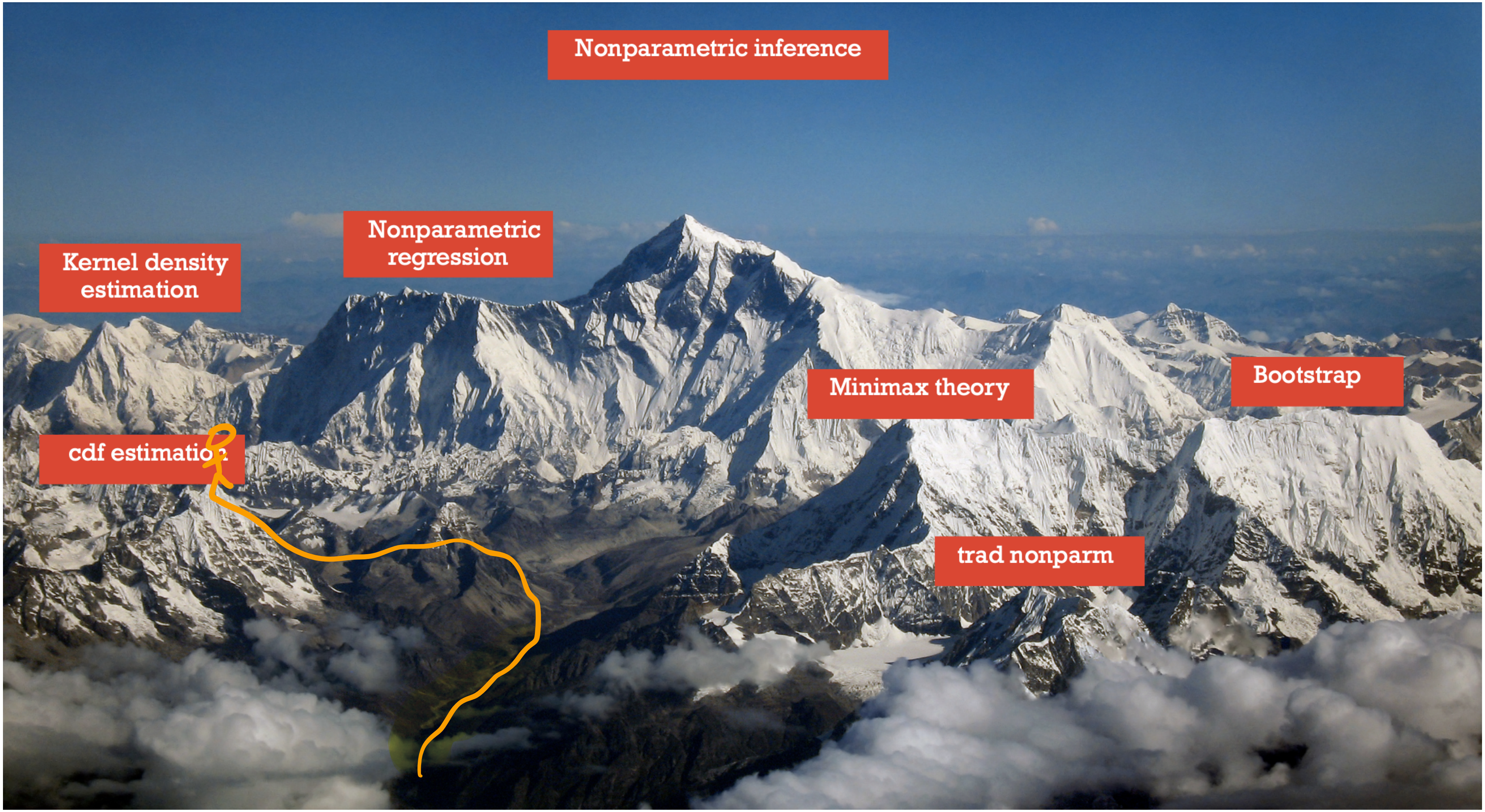
STAT 824 sp 2025 Lec 01 slides

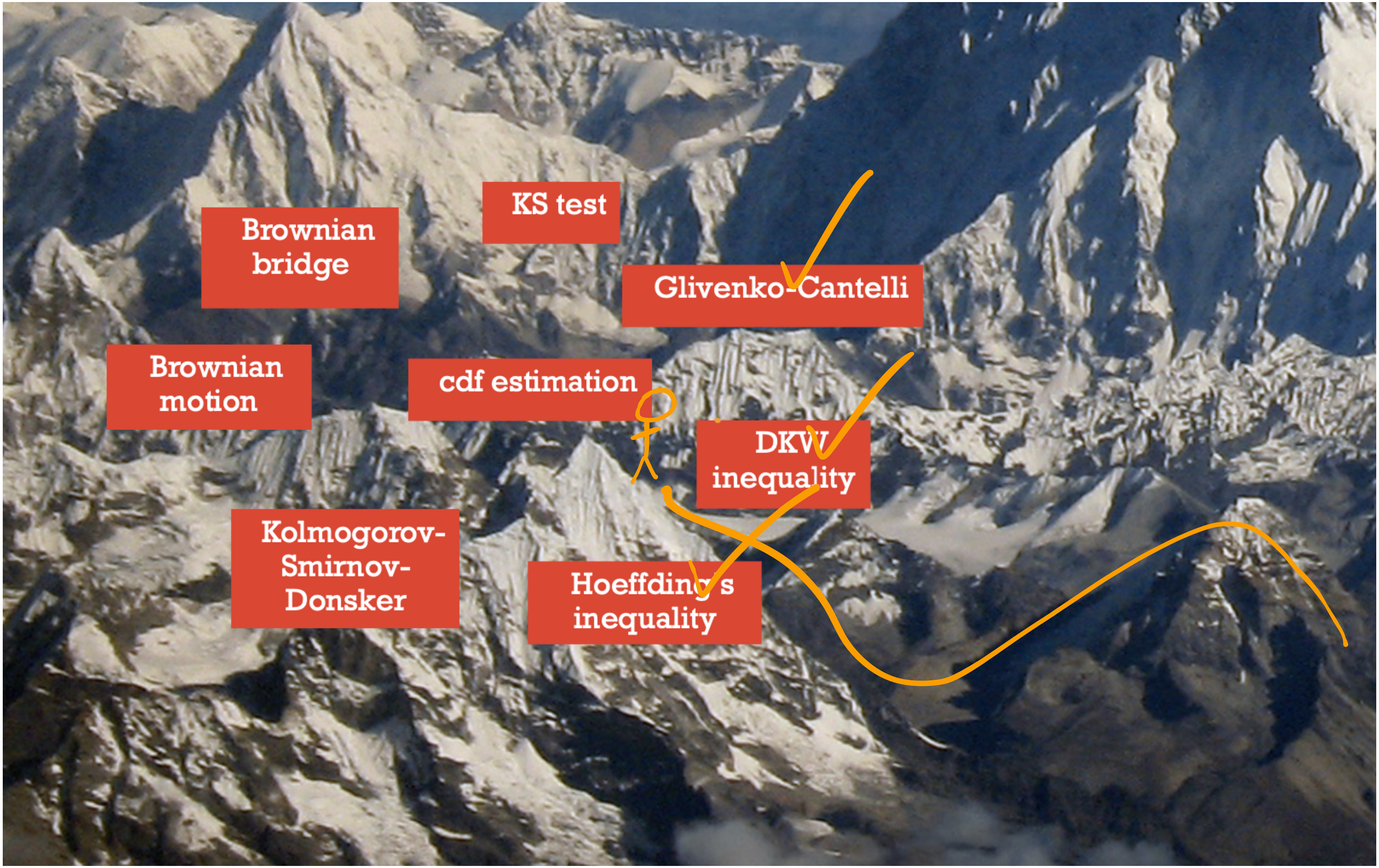
Estimating a cdf

Karl Gregory

University of South Carolina

These slides are an instructional aid; their sole purpose is to display, during the lecture, definitions, plots, results, etc. which take too much time to write by hand on the blackboard. They are not intended to explain or expound on any material.


Nonparametric inference
Nonparametric regression
Kernel density estimation
Minimax theory
Bootstrap
cdf estimation
trad nonparm


KS test
Brownian bridge
Glivenko-Cantelli
Brownian motion
cdf estimation
DKW inequality
Kolmogorov-Smirnov-Donsker
Hoeffding's inequality

Empirical cdf

The empirical cdf of a set of values $X_1, \ldots, X_n \in \mathbb{R}$ is given by

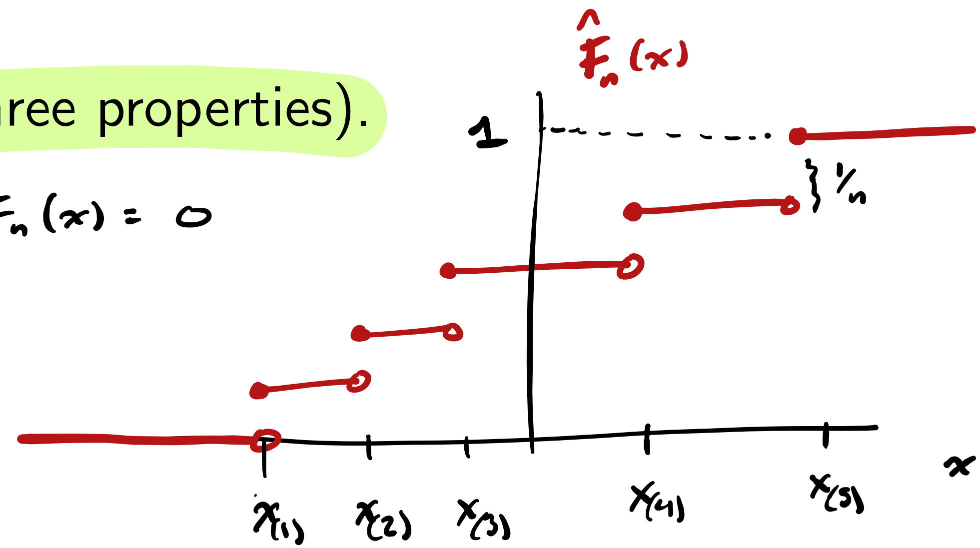
$$\hat{F}_n(x) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}(X_i \leq x) \quad \text{for all } x \in \mathbb{R}.$$

Discuss: Is this a legitimate cdf? (Three properties).

(i) $\lim_{x\to\infty} F_n(x) = 1 \qquad \lim_{x\to-\infty} F_n(x) = 0$

(ii) Nondecreasing

(iii) Right-continuous



Glivenko-Cantelli Theorem

If $X_1, \ldots, X_n$ is a rs from a distribution with cdf F,

$$\sup_{x \in \mathbb{R}} |\hat{F}_n(x) - F(x)| \longrightarrow 1$$

almost surely as $n \to \infty$.

Covered in STAT 810 and STAT 811.

random sample

Central limit result for empirical cdf at a point

If $X_1, \dots, X_n$ is a rs from a distribution with cdf F, then for each $x \in \mathbb{R}$ we have

$$\sqrt{n}(\hat{F}_n(x) - F(x)) \to \text{Normal}(0, F(x)[1 - F(x)]) \text{ in distribution}$$

as $n \to \infty$.

Exercise:

1. Prove the above result.
2. Use the result to construct an asymptotic $(1 - \alpha)100\%$ CI for $F(x)$.

① $\hat{F}_n(x) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}(X_i \le x) = \frac{1}{n}\sum_{i=1}^{n} Y_i$, $\quad Y_i = \mathbb{1}(X_i \le x)$

$Y_i \sim \text{Bernoulli}(F(x))$

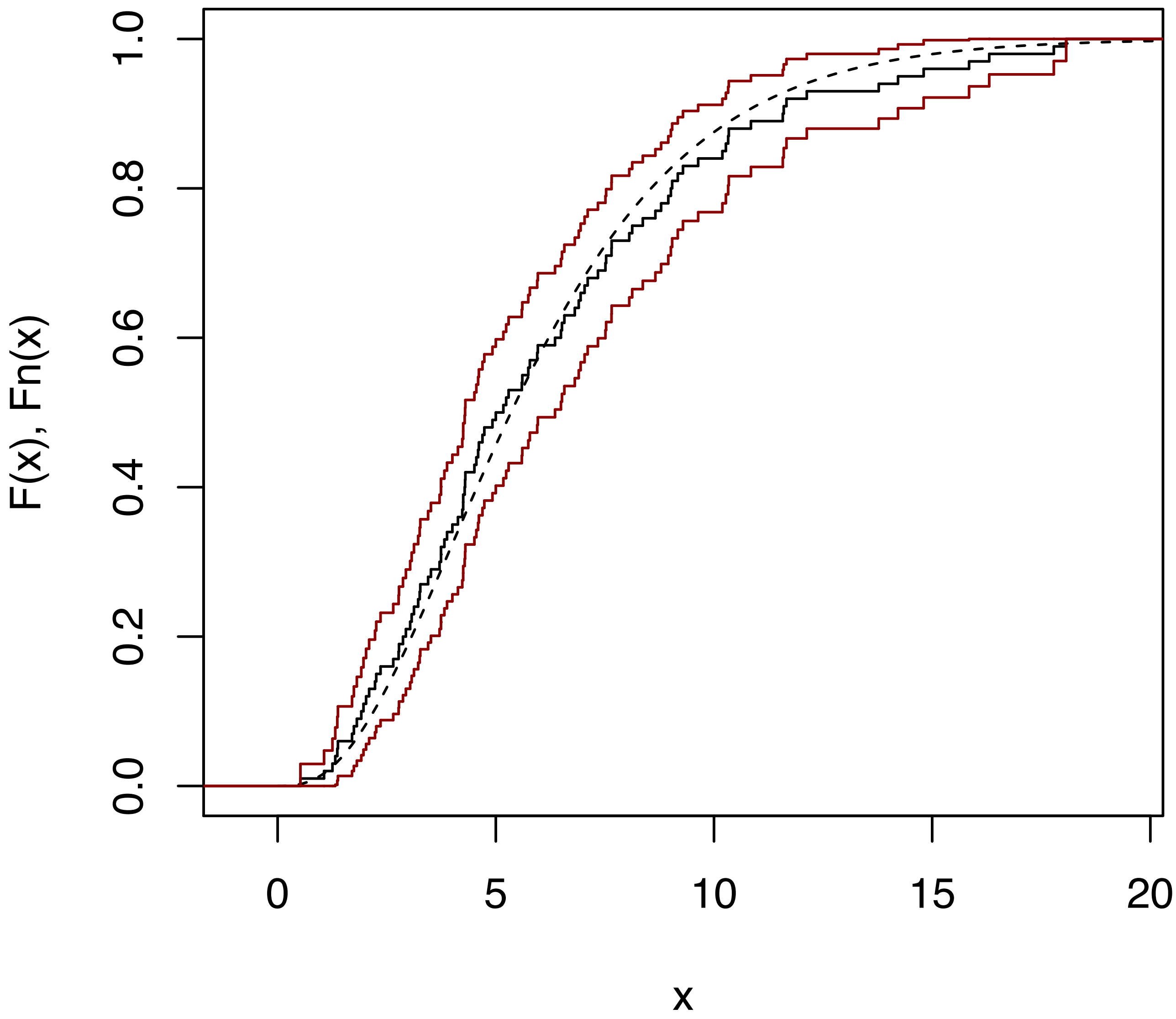
$\sqrt{n}(\hat{F}_n(x) - F(x)) = \sqrt{n}(\bar{Y}_n - \mathbb{E}\bar{Y}_n) \xrightarrow{D} \text{Normal}(0, F(x)[1-F(x)])$ as $n \to \infty$.

(2) $$\hat{F}_n(x) \pm z_{\alpha/2}\sqrt{\frac{\hat{F}_n(x)[1-\hat{F}_n(x)]}{n}}$$

Exercise: Generate some data $X_1, \ldots, X_n$ and make a plot with

1. the empirical cdf.
2. the true cdf.
3. pointwise confidence intervals at each of the values $X_1, \ldots, X_n$.

Can plot nicely with the `stepfun` function in R.


F(x), Fn(x)
0.0
0.2
0.4
0.6
0.8
1.0
0
5
10
15
20
x

Pointwise CIs versus confidence bands for a function

A $(1-\alpha) \times 100\%$

1. *confidence interval* for F at a point x is an interval $[L(x), U(x)]$ such that
$$P(L(x) \leq F(x) \leq U(x)) \geq 1 - \alpha.$$

2. *confidence band* for F over an interval $[a, b]$ is a region $\{(x, y) : L(x) \leq y \leq U(x), x \in [a, b]\}$ such that
$$P(L(x) \leq F(x) \leq U(x) \text{ for all } x \in [a, b]) \geq 1 - \alpha.$$

DKW

Dvoretzky-Kiefer-Wolfowitz inequality

If $X_1, \ldots, X_n$ is a rs from a distribution with cdf F, then for any $\varepsilon > 0$ we have

$$P\left(\sup_{x\in\mathbb{R}} |\hat{F}_n(x) - F(x)| \leq \varepsilon\right) \geq 1 - 2e^{-2n\varepsilon^2}.$$

Exercise:

1. Use the DKW result to construct a $(1-\alpha)\times 100\%$ confidence band for F.
2. Add the band to the plot with the pointwise CIs.

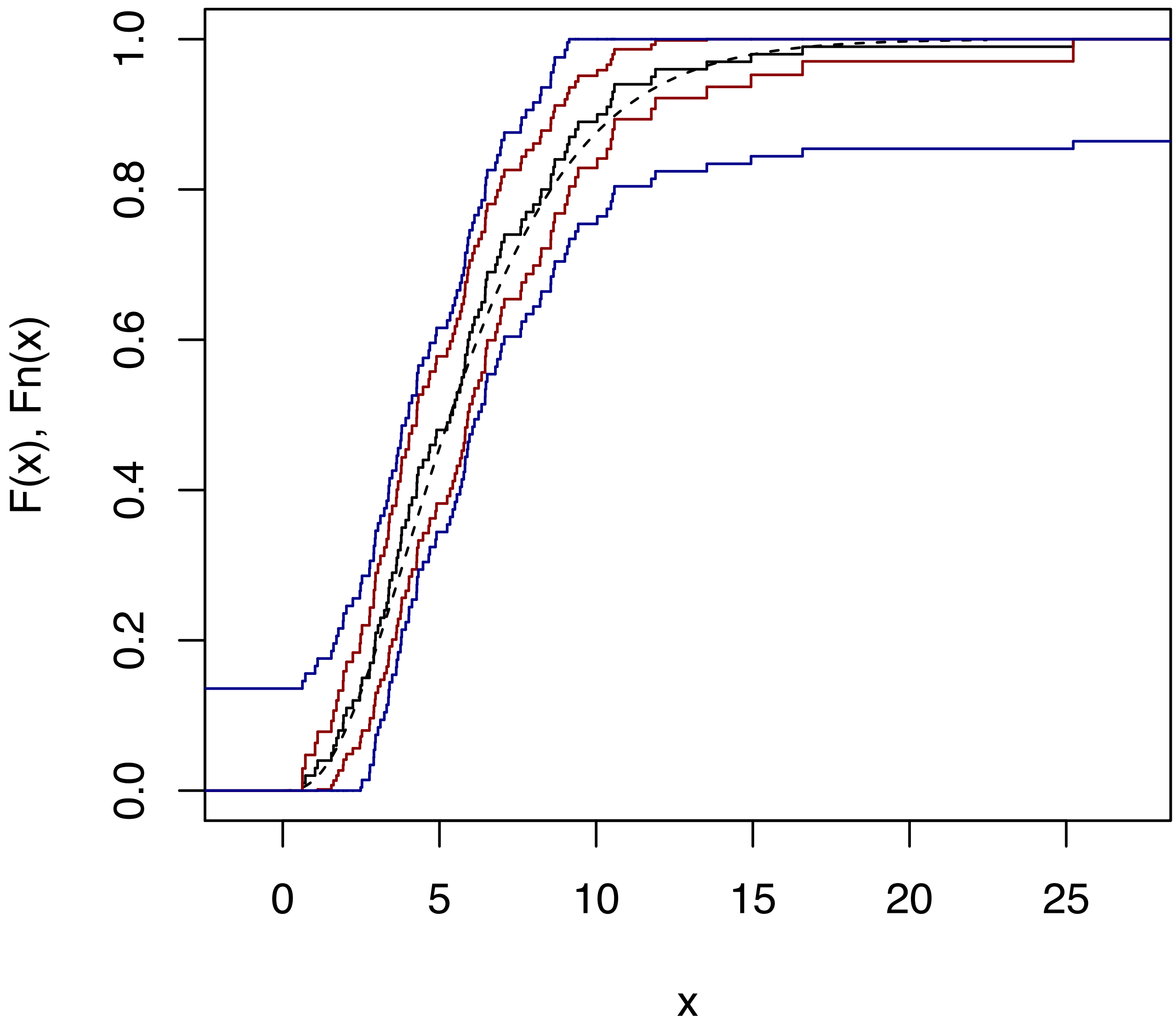
1

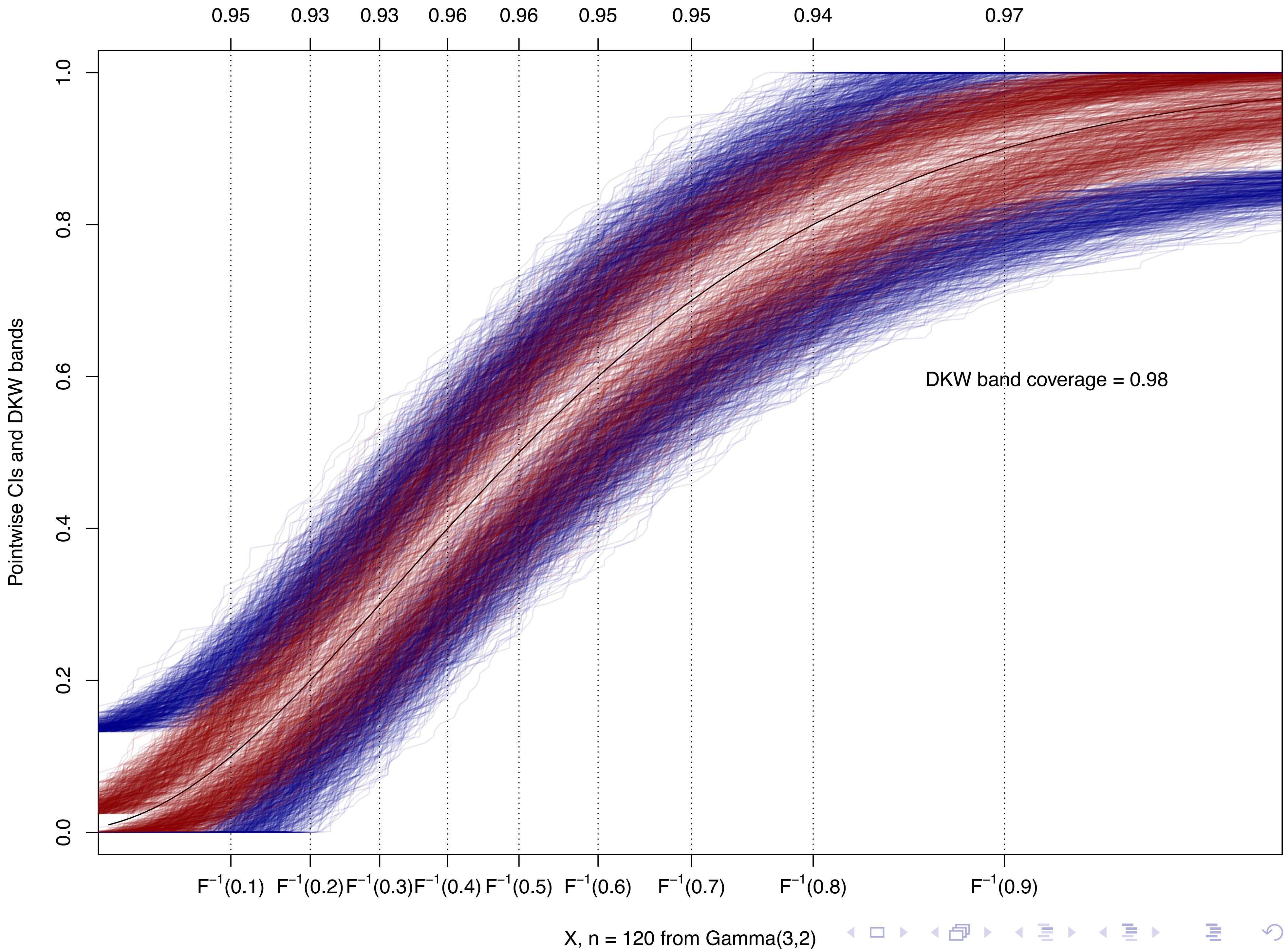
$$1 - 2e^{-2n\varepsilon^2} = 1 - \alpha$$

$$\iff \frac{\alpha}{2} = e^{-2n\varepsilon^2} \iff -2n\varepsilon^2 = \log\frac{\alpha}{2} \iff n\varepsilon^2 = \frac{1}{2}\log\left(\frac{2}{\alpha}\right)$$

$$\varepsilon = \sqrt{\frac{\log(2/\alpha)}{2n}}$$

Compute $\hat{F}_n(x) \pm \sqrt{\frac{\log(2/\alpha)}{2n}}$.


F(x), Fn(x)
0.0
0.2
0.4
0.6
0.8
1.0
0
5
10
15
20
25
x


0.95
0.93
0.93
0.96
0.96
0.95
0.95
0.94
0.97
1.0
0.8
0.6
0.4
0.2
0.0
Pointwise CIs and DKW bands
DKW band coverage = 0.98
$F^{-1}(0.1)$
$F^{-1}(0.2)$
$F^{-1}(0.3)$
$F^{-1}(0.4)$
$F^{-1}(0.5)$
$F^{-1}(0.6)$
$F^{-1}(0.7)$
$F^{-1}(0.8)$
$F^{-1}(0.9)$
X, n = 120 from Gamma(3,2)

Hoeffding's inequality can help us understand where DKW comes from:

Hoeffding's inequality

$\Rightarrow \mathbb{E} Y_i = 0$ $\quad a_i \le 0 \le b_i$

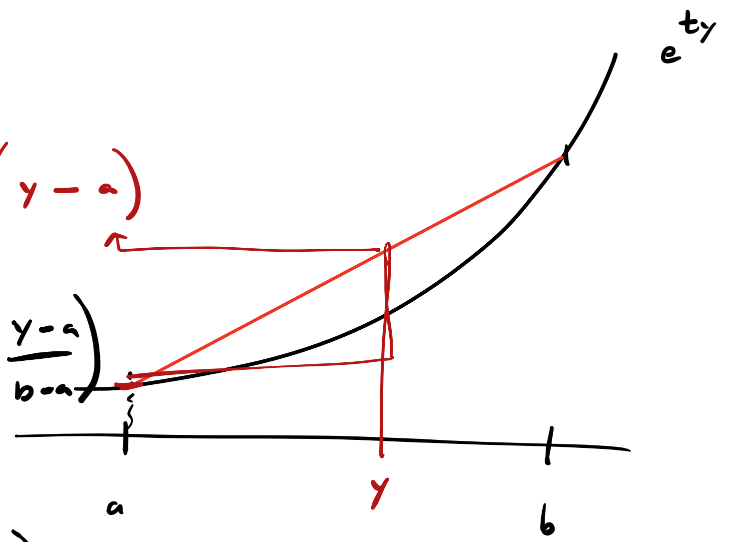
Let $Y_1, \ldots, Y_n$ be independent zero-mean rvs such that $Y_i \in [a_i, b_i]$, $i = 1, \ldots, n$. Then for any $\varepsilon > 0$ we have

$$P\Big(\sum_{i=1}^n Y_i \ge \varepsilon\Big) \le \exp\left(-\frac{2\varepsilon^2}{\sum_{i=1}^n (a_i - b_i)^2}\right).$$

Exercise:

1. For $Y \in [a, b]$ with zero mean, show that $\log \mathbb{E} e^{tY} \le t^2(b-a)^2/8$ for all t. ✓
2. Prove Hoeffding's inequality.

① For all $y \in [a,b]$



$$e^{ty} \leq e^{ta} + \left(\frac{e^{tb} - e^{ta}}{b-a}\right)(y-a)$$

$$= e^{tb}\left(\frac{y-a}{b-a}\right) + e^{ta}\left(1 - \frac{y-a}{b-a}\right)$$

$$= e^{tb}\left(\frac{y-a}{b-a}\right) + e^{ta}\left(\frac{b-y}{b-a}\right)$$

$$\mathbb{E}\, e^{tY} \leq e^{tb}\left(\frac{\mathbb{E}Y - a}{b-a}\right) + e^{ta}\left(\frac{b - \mathbb{E}Y}{b-a}\right) \qquad \mathbb{E}Y = 0.$$

$$= e^{ta}\left(\frac{b}{b-a}\right) - e^{tb}\left(\frac{a}{b-a}\right)$$

$$= e^{ta}\left[\frac{b}{b-a} - \frac{a}{b-a} e^{t(b-a)}\right]$$

$$\log \mathbb{E}\, e^{tY} \leq \underbrace{ta + \log\left(\frac{b}{b-a} - \frac{a}{b-a} e^{t(b-a)}\right)}_{=:\ \psi} \overset{\text{WTS}}{\leq} \frac{t^2(b-a)^2}{8}$$

For some $t > 0$,

$$\psi(t) = \underbrace{\psi(0)}_{=0} + \underbrace{\psi'(0)}_{=0}\, t + \frac{t^2}{2} \underbrace{\psi''(z)}_{\leq \frac{(b-a)^2}{4}}, \quad z \in [0,t]$$

$\vdots$

← At home

$$\leq \frac{(b-a)^2}{4}$$

$$\Rightarrow \quad \log \mathbb{E} e^{tY} \leq \frac{t^2 (b-a)^2}{8}$$

$$\mathbb{E} e^{tY} \leq e^{\frac{t^2 (b-a)^2}{8}}$$

X nonneg

$$P(X > \varepsilon) \leq \frac{\mathbb{E} X}{\varepsilon}$$

(Markov)

② Fix $\varepsilon > 0$: Then for any $t > 0$,

$$P\left(\sum_{i=1}^n Y_i \geq \varepsilon\right) = P\left(e^{t \sum_{i=1}^n Y_i} \geq e^{t\varepsilon}\right)$$

$$\leq \frac{\mathbb{E} e^{t \sum_{i=1}^n Y_i}}{e^{t\varepsilon}}$$

$$\overset{\text{(independence)}}{=} \frac{\prod_{i=1}^n \mathbb{E} e^{tY_i}}{e^{t\varepsilon}}$$

$$\leq \frac{\prod_{i=1}^n e^{t^2 \frac{(b_i - a_i)^2}{8}}}{e^{t\varepsilon}}$$

$$= e^{-t\varepsilon + t^2 \sum_{i=1}^n \frac{(b_i - a_i)^2}{8}}.$$

Minimize RHS in t. Result follows.

(3) $$P\Big(\sum_{i=1}^{n} Y_i \geq \varepsilon\Big) \leq \exp\left(-\frac{2\varepsilon^2}{\sum_{i=1}^{n}(a_i-b_i)^2}\right).$$

$$\sqrt{n}\left(\hat{F}_n(x) - F(x)\right) = \sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}\mathbb{I}(X_i \leq x) - F(x)\right)$$

$$= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\underbrace{\left[\mathbb{I}(X_i \leq x) - F(x)\right]}_{Y_i \in [\underset{a_i}{-F(x)},\ \underset{b_i}{1-F(x)}]}$$

$$= \frac{1}{\sqrt{n}}\sum_{i=1}^{n} Y_i$$

$\mathbb{E}\, Y_i = 0$

$b_i - a_i = 1$

Now

$$P\left(\hat{F}_n(x) - F(x) \geq \varepsilon\right) = P\left(\frac{1}{n}\sum_{i=1}^{n} Y_i \geq \varepsilon\right)$$

$$= P\left(\sum_{i=1}^{n} Y_i \geq n\varepsilon\right)$$

$$\overset{\text{Hoeffding's}}{\leq} \exp\left[-\frac{2(n\varepsilon)^2}{\underbrace{\sum_{i=1}^{n}(b_i - a_i)^2}}\right]$$

$$= e^{-2n\varepsilon^2}$$

* Not a proof of DKW, but you can see a connection

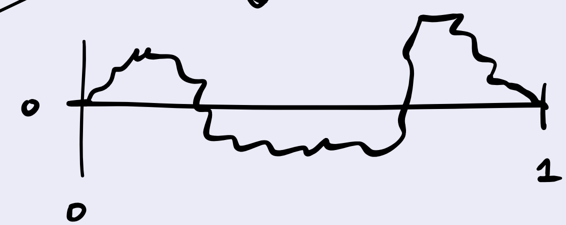
Kolmogorov-Smirnov-Donsker

If $X_1, \ldots, X_n$ is a rs from a distribution with *continuous* cdf F, then

1. $$\sqrt{n} \sup_{x \in \mathbb{R}} |\hat{F}_n(x) - F(x)| \to \sup_{t \in [0,1]} |B_0(t)| \quad \text{in distribution}$$

   looks like

   as $n \to \infty$, where B_0 is a *Brownian bridge*.



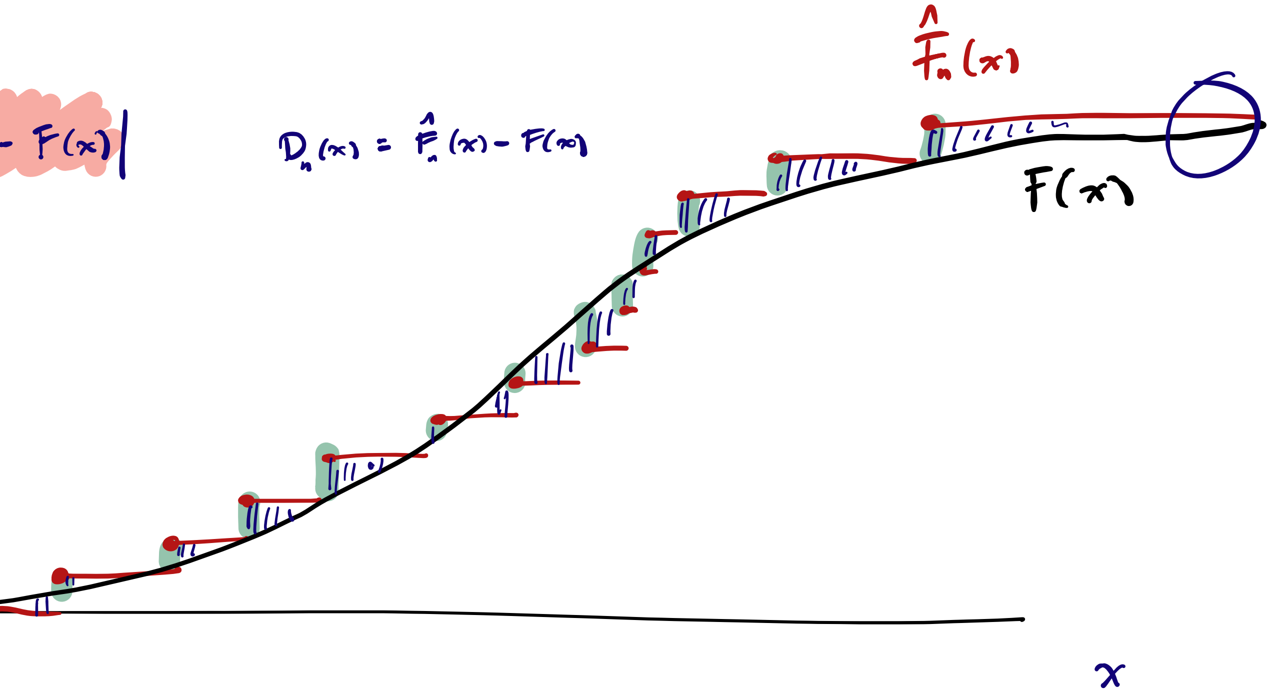
2. $$P\left(\sup_{t \in [0,1]} |B_0(t)| \leq x \right) = 1 - 2 \sum_{i=1}^{\infty} (-1)^{i+1} \exp(-2i^2x^2) \quad \text{for all } x \in \mathbb{R}.$$

Discuss: How to build confidence bands with above.

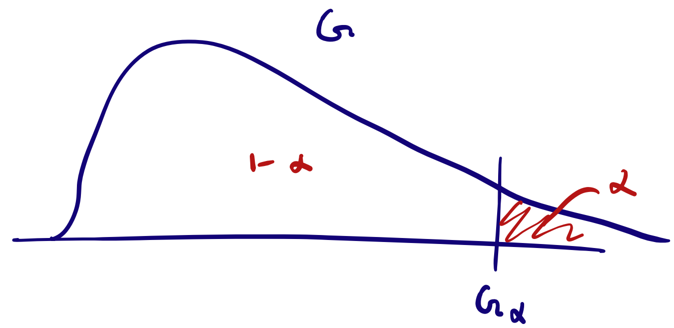
$$\sqrt{n}\ \sup_{x\in\mathbb{R}} \left|\hat{F}_n(x) - F(x)\right| \qquad D_n(x) = \hat{F}_n(x) - F(x)$$

$$\lim_{x\to\infty} F(x) = 1$$

$$\lim_{x\to\infty} \hat{F}_n(x) = 1$$



$$\sup_{t\in[0,1]} |B_0(t)| \sim G$$



Then

$$P\left(\sqrt{n}\ \sup_{x\in\mathbb{R}} \left|\hat{F}_n(x) - F(x)\right| \le G_\alpha \right) = 1-\alpha$$

Confidence band $\left[\hat{F}_n(x) \pm G_\alpha \frac{1}{\sqrt{n}} \right]$ for all x

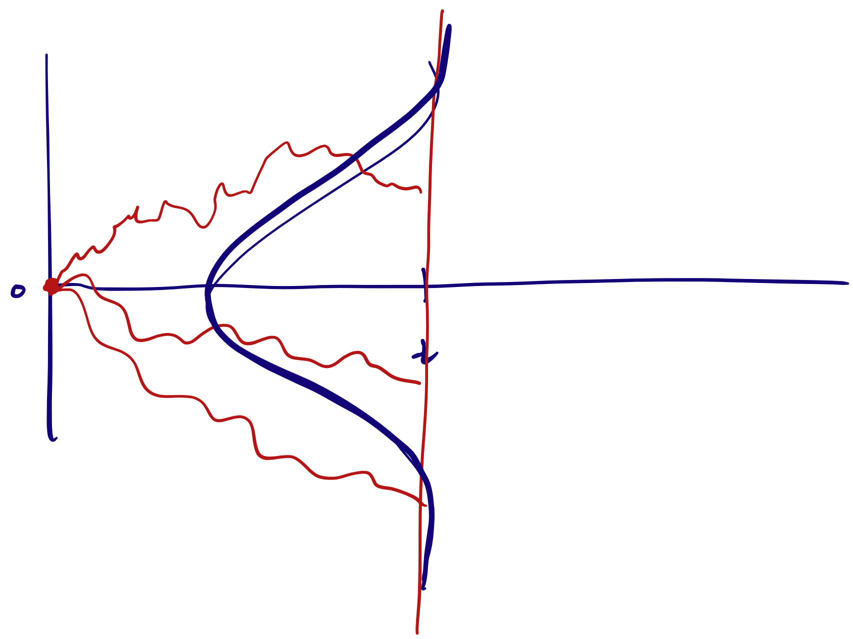
$$\sup_x \left|\hat{F}_n(x) - F(x)\right| = \max_{1\le i\le n} \left|\hat{F}_n(X_{(i)}) - F(X_i)\right|$$

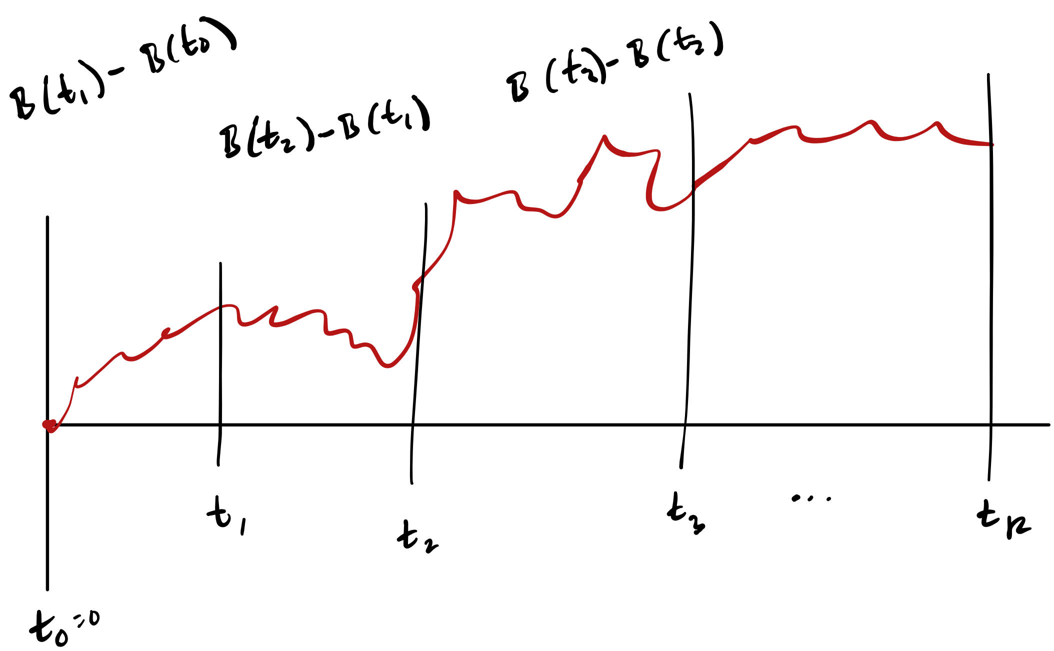
Brownian Motion / Wiener Process. Random function $B:[0,1]\to\mathbb{R}$

(i) $B(0)=0$

(ii) $B(t)\sim N(0,t)$

(iii) See below





random function

$C[0,1]$ space of continuous functions

Wiener process or standard Brownian motion

A *Wiener process* B is a rf in the space $C[0,1]$ of cont. fns on $[0,1]$ which satisfies

1. $B(0) = 0$ with probability 1.
2. $B(t) \sim \text{Normal}(0,t)$, for $t \in (0,1]$.
3. For $0 \leq t_0 \leq t_1 \leq \cdots \leq t_k \leq 1$, the increments

$$B(t_0) - B(0), \ldots, B(t_k) - B(t_{k-1})$$

are mutually independent.

This is also called *standard Brownian motion (SBM)*.

Simulate?

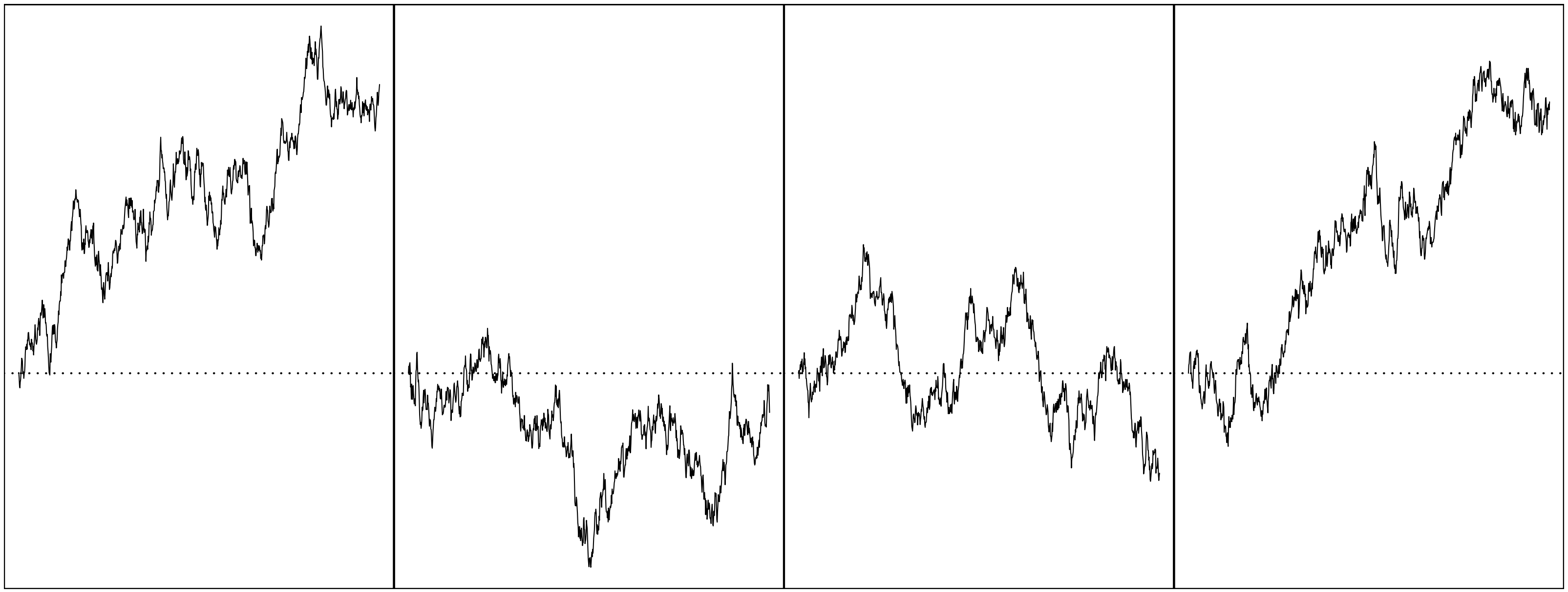
Generate an approximation to a standard Brownian motion

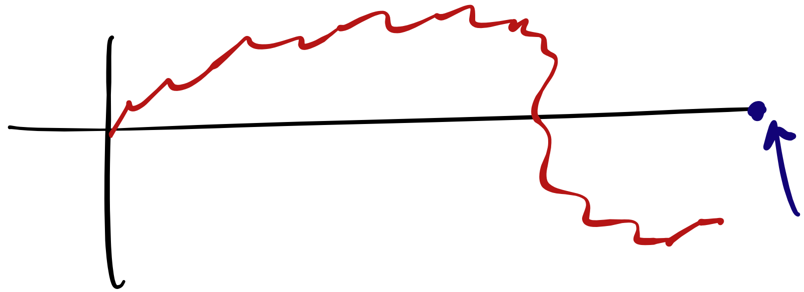
For each $n \geq 1$, let $B_n(0) = 0$, $B_n(\frac{1}{n}) = \frac{1}{\sqrt{n}} Z_1$ $B_n(\frac{2}{n}) = \frac{1}{\sqrt{n}}(Z_1 + Z_2)$ .....

$$B_n(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor tn \rfloor} Z_i, \quad Z_1, \dots, Z_n \overset{\text{ind}}{\sim} \text{Normal}(0, 1).$$

Then B_n converges to B as $n \to \infty$ by a functional CLT called Donsker's Theorem.

Exercise: Generate some (approximate) realizations of SBM and plot them.





Force it to end at zero.

Brownian bridge

A *Brownian bridge* is the random function in $C[0,1]$ given by

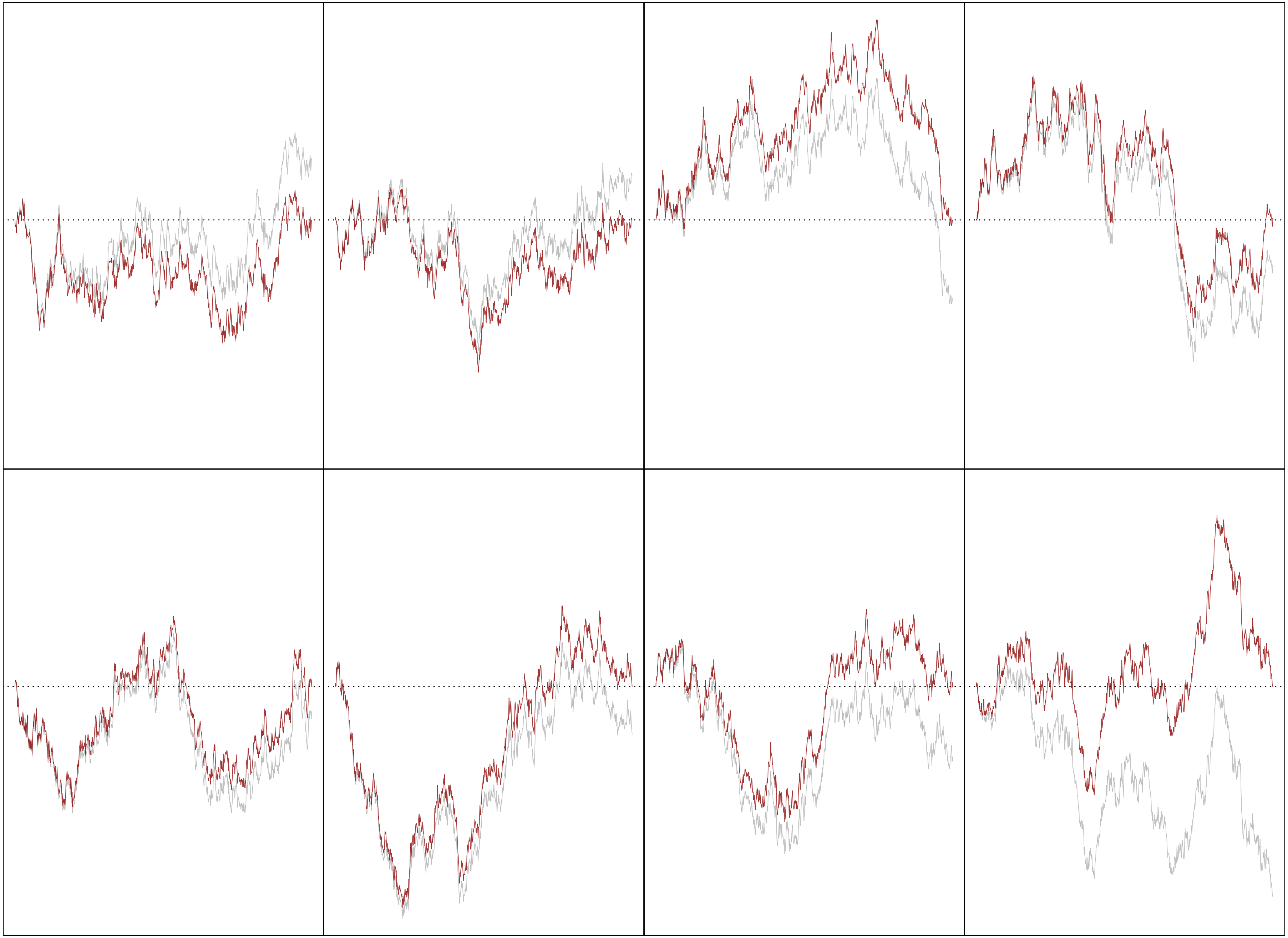
$$B_0(t) = B(t) - tB(1),$$

$t = 1$ $\quad B(1) - B(1)$

where B is a standard Brownian motion.

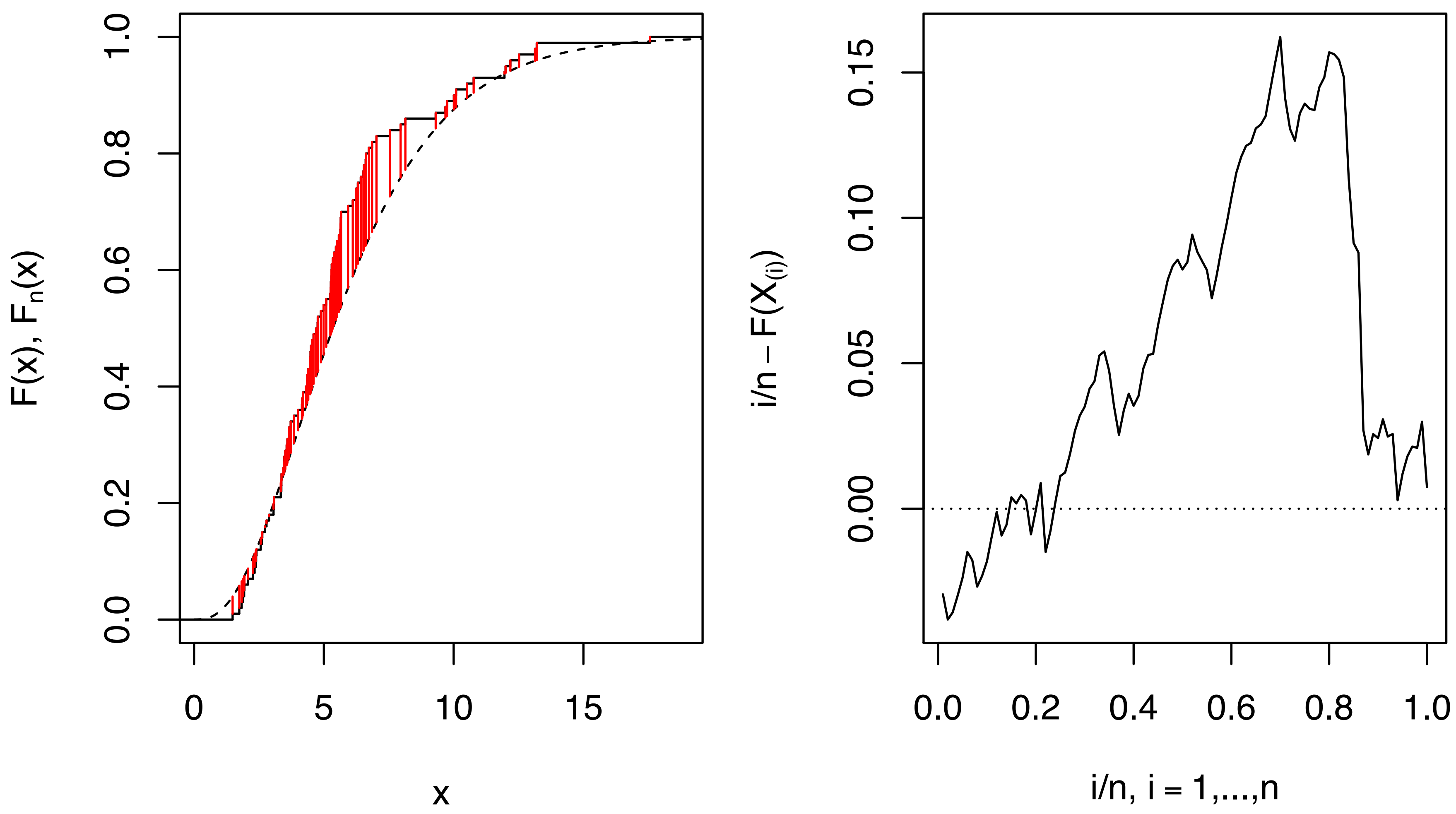
The "bridge" begins and ends at 0.

Exercise: Generate some (approximate) realizations of the Brownian bridge.



$\sqrt{n} \sup_{x \in \mathbb{R}} |\hat{F}_n(x) - F(x)| \xrightarrow{D} \sup_{t \in [0,1]} |B_0(t)|$

Basically, $\sqrt{n}[\hat{F}_n(X_{(i)}) - F(X_{(i)})]$, $i = 1, \ldots, n$, acts like a Br. bridge for large n.



DKW : $\hat{F}_n(x) \pm \sqrt{\frac{\log(2/.05)}{2n}}$ has coverage $\geq .95$ for all n.

KSD : $\hat{F}_n(x) \pm \frac{1.36}{\sqrt{n}}$ has coverage exactly .95 as $n \to \infty$.

Exercise:

≈ 1.36

1. Run a simulation to get the 0.95 quantile of $\sup_{t\in[0,1]} |B_0(t)|$.
2. Check accuracy using the cdf of $\sup_{t\in[0,1]} |B_0(t)|$.
3. Compute $\sqrt{[\log(2/0.05)]/2}$. From DKW inequality
4. Discuss.

1.358

$$P\left(\sup_{t\in[0,1]} |B_0(t)| \leq x\right) = 1 - 2\sum_{i=1}^{\infty}(-1)^{i+1}\exp(-2i^2x^2)$$

Let $X_1, \dots, X_n$ and $Y_1, \dots, Y_m$ be ind. rs with cdfs F and G, resp. Consider

$$H_0\colon F = G \text{ versus } H_1\colon F \neq G.$$

Two-sample Kolmogorov-Smirnov test

If $F = G$ the statistic

$$D_{nm} = \sup_{x \in \mathbb{R}} |\hat{F}_n(x) - \hat{G}_m(x)|$$

satisfies

$$P(\sqrt{mn/(m+n)} D_{nm} \leq x) \to 1 - 2 \sum_{i=1}^{\infty} (-1)^{i+1} e^{-2i^2 x^2}$$

as $n, m \to \infty$.

Compute D_{nm} as

$$D_{nm} = \max_{1 \leq i \leq n} [i/n - \hat{G}_m(X_{(i)})] \ \vee \ \max_{1 \leq j \leq m} [j/m - \hat{F}_n(Y_{(j)})].$$