STAT 824 sp 2025 Lec 09 slides

Edgeworth expansion and second-order correctness of the bootstrap

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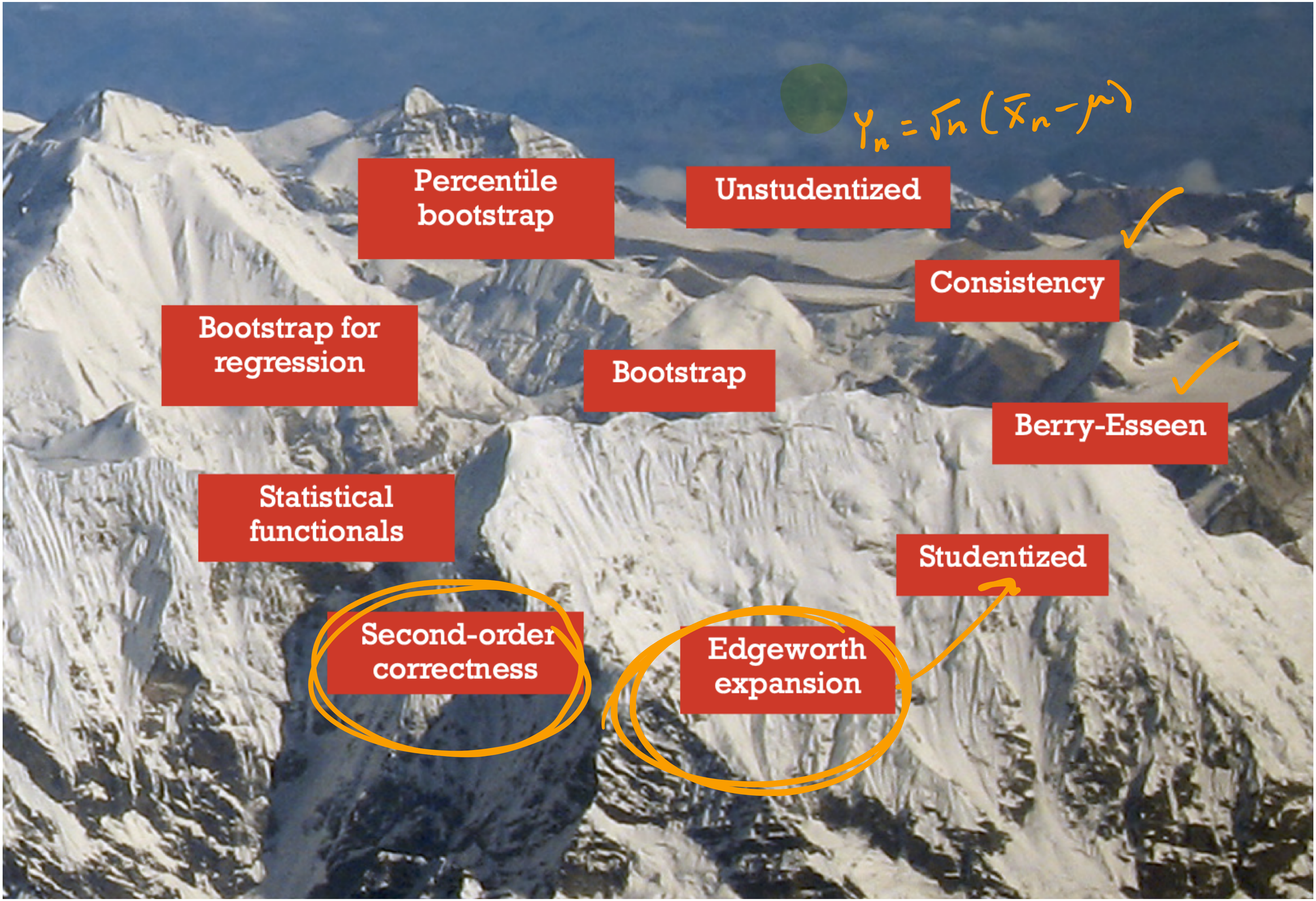
These slides are an instructional aid; their sole purpose is to display, during the lecture, definitions, plots, results, etc. which take too much time to write by hand on the blackboard. They are not intended to explain or expound on any material.


Percentile bootstrap
Unstudentized
$Y_n = \sqrt{n}(\bar{X}_n - \mu)$
Consistency
Bootstrap for regression
Bootstrap
Berry-Esseen
Statistical functionals
Studentized
Second-order correctness
Edgeworth expansion

Recall the central limit theorem:

For $X_1, \dots, X_n$ iid with mean μ and variance $\sigma^2 < \infty$,

$$Z_n = \sqrt{n}(\bar{X}_n - \mu)/\sigma \to \text{Normal}(0, 1)$$

in distribution as $n \to \infty$.

But we might want to know:

- How fast is the convergence?
- What features of the distribution of $X_1, \dots, X_n$ affect the rate and how?

Edgeworth expansions help us answer these questions.

$\mathbb{E}|X_1|^4 < \infty$

$\mathbb{E}|X_1|^3 < \infty$

Edgeworth expansion (2nd-order)

Let $X_1, \dots, X_n$ be iid with $\mathbb{E}X_1 = \mu$, $\text{Var}\, X_1 = \sigma^2 \in (0, \infty)$, and $\mathbb{E}|X_1|^{j+2} < \infty$ for $j = 1, 2$, and set $Z_n = \sqrt{n}(\bar{X}_n - \mu)/\sigma$. Then if $\limsup_{|t|\to\infty} |\mathbb{E}\exp(\iota t X_1)| < 1$ (Cramer's condition) we have

Expansion of order 0.

$$P(Z_n \leq x) = \boxed{\Phi(x)} + n^{-1/2} p_1(x)\phi(x) + n^{-1} p_2(x)\phi(x) + o(n^{-1})$$

as $n \to \infty$, where $p_1(x)$ and $p_2(x)$ are given by

$H_2(x)$

"kurtosis"

$H_5(x)$

$H_3(x)$

$$p_1(x) = \frac{1}{6}\boxed{\frac{\mu_3}{\sigma^3}}(x^2 - 1)$$

$$p_2(x) = \frac{1}{24}\left(\boxed{\frac{\mu_4}{\sigma^4}} - 3\right)(x^3 - 3x) + \frac{1}{72}\boxed{\frac{\mu_3^2}{\sigma^6}}(x^5 - 10x^3 + 15x).$$

In the above $\mu_3 = \mathbb{E}(X_1 - \mu)^3$ and $\mu_4 = \mathbb{E}(X_1 - \mu)^4$.

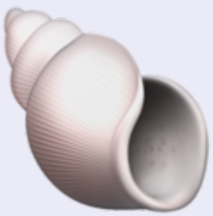
Discuss: The role of moments in the Edgeworth expansions.

To derive the Edgeworth expansions, we will need several tools, starting with...

Hermite polynomials

The *Hermite polynomials* $H_1, H_2, \ldots$ are defined by the relation

$$(-1)^k \frac{d^k}{dx^k}\phi(x) = H_k(x)\phi(x), \quad k = 1, 2, \ldots$$



Exercise: Find the first 3 Hermite polynomials.

$\underline{k=1}$ $\quad (-1)\frac{d}{dx}\phi(x) = (-1)\frac{d}{dx}\frac{1}{\sqrt{2\pi}}e^{-x^2/2} = (-1)\frac{1}{\sqrt{2\pi}}e^{-x^2/2}(-x) = x\phi(x)$

$H_1(x) = x.$

$H_2(x) = x^2 - 1$ $\quad H_3(x) = x^3 - 3x$, $\quad H_4(x) = x^4 - 6x^2 + 3$

$H_5(x) = x^5 - 10x^3 + 15$

Inversion formula

If X is a rv with ch. function ψ_X such that $\int_{-\infty}^{\infty} |\psi_X(t)| dt < \infty$, then X has pdf

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-\iota tx)\psi_X(t) dt \quad \text{for all } x \in \mathbb{R}.$$

Exercise: Use the inversion formula to establish the useful identity

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\iota tx} e^{-t^2/2} (\iota t)^k dt = H_k(x)\phi(x).$$

characteristic function of $N(0,1)$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\iota tx} e^{-t^2/2} (\iota t)^k dt = (-1)^k \frac{d^k}{dx^k} \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\iota tx} e^{-t^2/2} dt}_{\phi(x)} = (-1)^k \frac{d^k}{dx^k}\phi(x) = H_k(x)\phi(x)$$

$$\frac{d^k}{dx^k} e^{-\imath t x} = e^{-\imath t x} (-\imath t)^k = e^{-\imath t x} (\imath t)^k (-1)^k$$

$$\Rightarrow (-1)^k \frac{d^k}{dx^k} e^{-\imath t x} = e^{-\imath t x} (\imath t)^k$$

Let $X_1, \dots, X_n$ have mean $\mu = 0$, $\sigma^2 = 1$,

$$\mathbb{E} X_1^3 = \underset{\uparrow \atop \frac{\mu_3}{\sigma_3}}{\delta}, \quad \mathbb{E} X_1^4 = \underset{\uparrow \atop \frac{\mu_4}{\sigma_4}}{\tau}.$$

$$Z_n = \frac{\sqrt{n}\,(\bar{X}_n - \mu)}{\sigma} = \sqrt{n}\,\bar{X}_n$$

characteristic fcn.

$$\psi_{Z_n}(t) = \mathbb{E} \exp(\imath t Z_n)$$

$$= \mathbb{E} \exp\left[\imath t \sqrt{n}\, \bar{X}_n\right]$$

$$= \mathbb{E} \exp\left[\imath t \frac{1}{\sqrt{n}} \sum_{i=1}^n X_i\right]$$

$$= \prod_{i=1}^n \mathbb{E} \exp\left[\imath t/\sqrt{n}\; X_i\right]$$

$$= \left(\mathbb{E} \exp\left[\imath t/\sqrt{n}\; X_1\right]\right)^n$$

Taylor expand around $t = 0$

$$\mathbb{E} \exp\left[it/\sqrt{n} \; X_1 \right] = \mathbb{E}\left[1 + \frac{(it)}{\sqrt{n}} X_1 + \frac{(it)^2}{2n} X_1^2 + \frac{(it)^3}{6 n^{3/2}} X_1^3 + \frac{(it)^4}{24 n^2} X_1^4 + o(n^{-2}) \right]$$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots$$

$$= 1 - \frac{t^2}{2n} + \frac{(it)^3}{6 n^{3/2}} \gamma + \frac{(it)^4}{24 n^{-2}} \tau + o(n^{-2})$$

$$(a+b)^n = \sum_{i=0}^{n} a^i b^{n-i} \binom{n}{i}$$

$$(a_1 + \dots + a_m)^n = \sum_{\substack{n_1 + \dots + n_m = n \\ n_1, \dots, n_m \in \{0, \dots, n\}}} \left(\frac{n!}{n_1! \cdots n_m!} \right) a_1^{n_1} \cdot \dots \cdot a_m^{n_m}$$

$$\psi_{Z_n}(t) = \left[\left(1 - \frac{t^2}{2n}\right) + \frac{(it)^3}{6 n^{3/2}} \gamma + \frac{(it)^4}{24 n^{-2}} \tau + o(n^{-2}) \right]^n$$

$$= \underbrace{\left(1 - \frac{t^2}{2n}\right)^n}_{\approx e^{-t^2/2}} + \underbrace{\left(1 - \frac{t^2}{2n}\right)^{n-1}} \frac{(it)^3 \gamma}{6 n^{3/2}} n$$

for large n

$$+ \underbrace{\left(1 - \frac{t^2}{2n}\right)^{n-1}} \frac{(it)^4 \tau}{24 n^{-2}} n$$

$$+ \underbrace{\left(1 - \frac{t^2}{2n}\right)^{n-2}} \left(\frac{(it)^3}{6 n^{3/2}} \gamma \right)^2 \frac{n(n-1)}{2} + o(n^{-1}) \Big]$$

(Use fact: $\left(1+\frac{a}{n}\right)^{n-k} = e^{a}\left(1 - \frac{a(a+k)}{2n}\right) + o(n^{-1})$)

$\vdots$

$$\psi_{Z_n}(t) = e^{-t^2/2}\left[1 - \frac{\gamma}{6\sqrt{n}}(\iota t)^3 + \frac{(\tau-3)}{24n}(\iota t)^4 + \frac{\gamma^2(\iota t)^6}{72n}\right] + o(n^{-1})$$

$$\tilde{\psi}_{Z_n}(t) = e^{-t^2/2}\left[1 - \frac{\gamma}{6\sqrt{n}}(\iota t)^3 + \frac{(\tau-3)}{24n}(\iota t)^4 + \frac{\gamma^2(\iota t)^6}{72n}\right]$$

Now use inversion formula:

$$f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-\underset{\sqrt{-1}}{\iota} tx)\psi_X(t)dt$$

$$\tilde{f}_{Z_n}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\iota tx}\, e^{-t^2/2}\left[1 - \frac{\gamma}{6\sqrt{n}}(\iota t)^3 + \frac{(\tau-3)}{24n}(\iota t)^4 + \frac{\gamma^2(\iota t)^6}{72n}\right]dt$$

$$= \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\iota tx}\, \underbrace{\boxed{e^{-t^2/2}}}_{\text{characteristic function of } \phi}\, dt}_{\phi(x)} - \frac{\gamma}{6\sqrt{n}}\underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-t^2/2}\, e^{-\iota tx}(\iota t)^3\, dt}_{H_3(x)\,\phi(x)}$$

$$+ \frac{(\tau-3)}{24n}\underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-t^2/2}\, e^{-\iota tx}(\iota t)^4\, dt}_{H_4(x)\,\phi(x)} + \frac{\gamma^2}{72n}\underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-t^2/2}\, e^{-\iota tx}(\iota t)^6\, dt}_{H_6(x)\,\phi(x)}$$

$$= \phi(x) - \frac{\sigma}{6\sqrt{n}} H_3(x)\,\phi(x) + \frac{(\tau-3)}{24n} H_4(x)\,\phi(x) + \frac{\sigma^2}{72n} H_6(x)\,\phi(x)$$

$$\tilde{F}_{Z_n}(x) = \int_{-\infty}^{x} \tilde{f}_{Z_n}(t)\,dt$$

$$= \Phi(x) + \frac{\sigma}{6\sqrt{n}} \underbrace{H_2(x)}_{(x^2-1)}\,\phi(x) - \frac{(\tau-3)}{24n} \underbrace{H_3(x)}_{(x^3-3x)}\,\phi(x) - \frac{\sigma^2}{72n} \underbrace{H_5(x)}_{x^5-10x^3+15x}\,\phi(x)$$

$X_1, \dots, X_n \overset{iid}{\sim} F$ with mean μ and variance $\sigma^2 < \infty$.

$$Y_n = \sqrt{n}\,(\bar{X}_n - \mu) \xrightarrow{D} N(0, \sigma^2)$$

$$T_n = \frac{\sqrt{n}\,(\bar{X}_n - \mu)}{S_n} \xrightarrow{D} N(0,1)$$

A $(1-\alpha)^* 100\%$ asymptotic C.I. for μ is

$$\bar{X}_n \pm z_{\alpha/2} \frac{S_n}{\sqrt{n}}$$

Exercise: Let $X_1, \dots, X_n$ be iid with

$$\mathbb{E}X_1 = 0, \quad \mathbb{E}X_1^2 = 1, \quad \mathbb{E}X_1^3 = \gamma, \quad \mathbb{E}X_1^4 = \tau$$

and $|\psi_X(t)| < 1$ for all $t \neq 0$, where $\psi_X(t) = \mathbb{E}\exp(\iota t X_1)$.

Sketch the proof of the Edgeworth expansion for $\sqrt{n}\bar{X}_n$ in these steps:

1. Write the characteristic function of $S_n = \sqrt{n}\bar{X}_n$ as $[\psi_X(t/\sqrt{n})]^n$.
2. Taylor expand $\psi_X(t/\sqrt{n})$ around $t = 0$.
3. Raise expansion to power n, discarding terms of order $o(n^{-1})$.
4. Make use of this fact: For each nonnegative integer k,

$$\left(1 + \frac{a}{n}\right)^{n-k} = e^a\left[1 - \frac{a(a+k)}{2n}\right] + o(n^{-1}) \quad \text{as } n \to \infty.$$

5. Again discard terms of order $o(n^{-1})$ to get approximation $\tilde{\psi}_{S_n}$ to ψ_{S_n}.
6. Use inversion formula to invert $\tilde{\psi}_{S_n}$ into the corresponding pdf $\tilde{f}_{S_n}$.
7. Take the antiderivative of $\tilde{f}_{S_n}$ using $\frac{d}{dx}[H_k(x)\phi(x)] = -H_{k+1}(x)\phi(x)$.

$\mathbb{E}\, X_1 = \lambda$

$\mathrm{Var}\, X_1 = \lambda^2$

$Z_n = \sqrt{n}(\bar{x}_n - \mu)/\sigma$

Example: Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Exponential}(\lambda)$. Then $Z_n = \sqrt{n}(\bar{X}_n - \lambda)/\lambda$ has density given by

$$f_{Z_n}(z) = \frac{1}{\Gamma(n)(1/\sqrt{n})^n}(z + \sqrt{n})^{n-1} \exp\left(-\frac{z + \sqrt{n}}{1/\sqrt{n}}\right)(z > -\sqrt{n}).$$

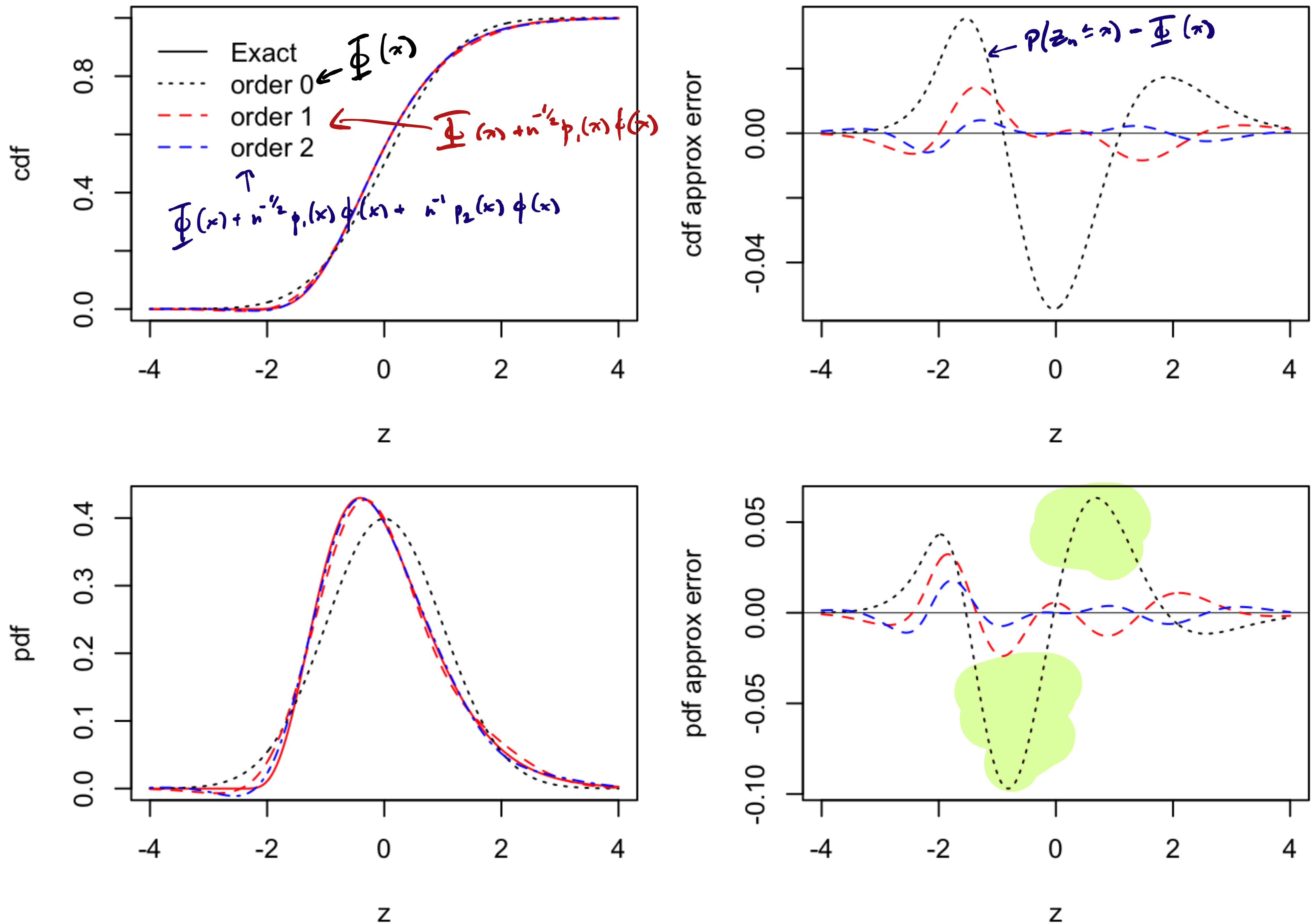
Moreover $\mu_3/\sigma^3 = 2$ and $\mu_4/\sigma^4 = 9$. One may compare the Edgeworth expansions with the exact sampling distribution of Z_n.

$\frac{\sqrt{n}\,\bar{X}_n}{\lambda} \sim \text{Gamma}\left(n, \frac{1}{\sqrt{n}}\right)$

$\mathbb{E}\, \frac{\sqrt{n}\,\bar{X}_n}{\lambda} = n \frac{1}{\sqrt{n}} = \sqrt{n}$

$\mathrm{Var}\left(\frac{\sqrt{n}\,\bar{X}_n}{\lambda}\right) = n\left(\frac{1}{\sqrt{n}}\right)^2 = 1.$

$Z_n = \frac{\sqrt{n}(\bar{x}_n - \lambda)}{\lambda} = \frac{\sqrt{n}\,\bar{X}_n}{\lambda} - \sqrt{n}$


Exact
order 0
order 1
order 2
$\Phi(x)$
$\Phi(x) + n^{-1/2} p_1(x)\phi(x)$
$\Phi(x) + n^{-1/2} p_1(x)\phi(x) + n^{-1} p_2(x)\phi(x)$
cdf
0.0
0.4
0.8
-4
-2
0
2
4
z
$P(Z_n \le x) - \Phi(x)$
cdf approx error
0.00
-0.04
pdf
0.0
0.1
0.2
0.3
0.4
pdf approx error
0.05
0.00
-0.05
-0.10

Edgeworth expansion for studentized pivot (2nd-order)

Let $X_1, \ldots, X_n$ be iid with $\mathbb{E}X_1 = \mu$, $\operatorname{Var} X_1 = \sigma^2 \in (0, \infty)$, and $\mathbb{E}|X_1|^{j+2} < \infty$ for $j = 1, 2$, and set $T_n = \sqrt{n}(\bar{X}_n - \mu)/S_n$. Then if $\limsup_{|t|\to\infty} |\mathbb{E}\exp(\iota t X_1)| < 1$ (Cramer's condition) we have

$$P(T_n \leq x) = \Phi(x) + n^{-1/2} q_1(x)\phi(x) + n^{-1} q_2(x)\phi(x) + o(n^{-1})$$

as $n \to \infty$, where

$$q_1(x) = \frac{1}{6}\frac{\mu_3}{\sigma^3}(2x^2 + 1)$$

$$q_2(x) = \frac{1}{12}(\frac{\mu_4}{\sigma^4} - 3)(x^3 - 3x) - \frac{1}{18}\frac{\mu_3^2}{\sigma^6}(x^5 + 2x^3 - 3x) - \frac{1}{4}(x^3 + 3x).$$

Remember our pivot quantities for the mean:

$$Y_n = \sqrt{n}(\bar{X}_n - \mu) \quad \text{and} \quad T_n = \frac{\sqrt{n}(\bar{X}_n - \mu)}{S_n}.$$

Accuracies of Normal approximations to pivot distributions

Letting G_{Y_n} and G_{T_n} be the cdfs of these pivots, we obtain

$$\sup_{x\in\mathbb{R}} |G_{Y_n}(x) - \Phi(x/\sigma)| = O(n^{-1/2})$$

$$\sup_{x\in\mathbb{R}} |G_{T_n}(x) - \underline{\Phi(x)}| = O(n^{-1/2}).$$

as $n \to \infty$ from the Edgeworth expansions.

So the error of the Normal approximation to these cdfs is of the order $O(n^{-1/2})$.

Allows 1st-order Edgeworth

Edgeworth expansion results for the bootstrap

Let $X_1, \dots, X_n$ be iid, $\mathbb{E}X_1 = \mu$, $\operatorname{Var} X_1 = \sigma^2 \in (0, \infty)$, $\mathbb{E}|X_1|^3 < \infty$.

Then under Cramer's condition we have

Bootstrap estimate

$$\text{1} \quad \sup_{x \in \mathbb{R}} |\hat{G}_{Y_n}(x) - G_{Y_n}(x)| = O(n^{-1/2})$$

$$\text{2} \quad \sup_{x \in \mathbb{R}} |\hat{G}_{T_n}(x) - G_{T_n}(x)| = O(n^{-1})$$

almost surely as $n \to \infty$.

See first equation on pg. 240 of [1] following Theorem 5.1.

The bootstrap estimators $\hat{G}_{T_n}(x)$ is *second-order correct*.

Exercise: Sketch the proof of each of the results above.

(1) $\sup_{x\in\mathbb{R}} |\hat{G}_{Y_n}(x) - G_{Y_n}(x)| = O(n^{-1/2})$

$$
\begin{aligned}
G_{Y_n}(x) &= P(Y_n \le x) \\
&= P(\sqrt{n}(\bar{X}_n - \mu) \le x) \\
&= P(\sqrt{n}(\bar{X}_n - \mu)/\sigma \le x/\sigma) \\
&= P(Z_n \le x/\sigma) \\
&= \Phi(x/\sigma) + \frac{1}{\sqrt{n}} \frac{1}{6} \frac{\mu_3}{\sigma^3} (x^2 - 1)\phi(x) + O(n^{-1})
\end{aligned}
$$

$P(Z_n \le x) = \boxed{\Phi(x)} + n^{-1/2} p_1(x)\phi(x) + n^{-1} p_2(x)\phi(x) + o(n^{-1})$ ← Expansion

$n \to \infty$, where $p_1(x)$ and $p_2(x)$ are given by

$$p_1(x) = \frac{1}{6}\boxed{\frac{\mu_3}{\sigma^3}}\underbrace{(x^2 - 1)}_{H_2(x)}$$

"kurtosis"

$$p_2(x) = \frac{1}{24}\left(\boxed{\frac{\mu_4}{\sigma^4}} - 3\right)\underbrace{(x^3 - 3x)}_{H_3(x)} + \frac{1}{72}\boxed{\frac{\mu_3^2}{\sigma^6}}\underbrace{(x^5 - 10x^3 + 15x)}_{H_5(x)}.$$

$$
\begin{aligned}
\hat{G}_{Y_n}(x) &= P_*(Y_n^* \le x) \\
&= P_*(\sqrt{n}(\bar{X}_n^* - \bar{X}_n) \le x) \\
&= P_*\left(\frac{\sqrt{n}(\bar{X}^* - \bar{X}_n)}{\hat{\sigma}_n} \le x/\hat{\sigma}_n\right) \\
&= P_*(Z_n^* \le x/\hat{\sigma}_n)
\end{aligned}
$$

(If can be shown) (a.s.)

$$= \Phi(x/\hat{\sigma}_n) + \frac{1}{\sqrt{n}} \frac{1}{6} \frac{\hat{\mu}_3}{\hat{\sigma}^3} (x^2 - 1)\phi(x) + O(n^{-1})$$

$\Rightarrow$

$$\sup_{x\in\mathbb{R}} \left| \hat{G}_{Y_n}(x) - G_{Y_n}(x) \right|$$

$$\leq \sup_{x \in \mathbb{R}} \underbrace{\left| \Phi\left(\frac{x}{\sigma}\right) - \Phi\left(x/\hat{\sigma}_n\right) \right|}_{O(n^{-1/2}) \text{ a.s.}}$$

$$+ \underbrace{\frac{1}{\sqrt{n}} \frac{1}{6} \underbrace{\left| \frac{\mu_3}{\sigma^3} - \frac{\hat{\mu}_3}{\hat{\sigma}^3} \right|}_{\to 0 \text{ a.s.}}}_{o(n^{-1/2})} \sup_{x \in \mathbb{R}} \left| (x^2 - 1)\phi(x) \right| + O(n^{-1})$$

$$= O(n^{-1/2}).$$

$$P(T_n \leq x) = \Phi(x) + n^{-1/2} q_1(x)\phi(x) + \underbrace{n^{-1} q_2(x)\phi(x) + o(n^{-1})}_{O(n^{-1})}$$

as $n \to \infty$, where

$$q_1(x) = \frac{1}{6}\frac{\mu_3}{\sigma^3}(2x^2 + 1)$$

(2)

$$G_{T_n}(x) = P(T_n \leq x)$$

$$= \Phi(x) + \frac{1}{\sqrt{n}} \frac{1}{6} \frac{\mu_3}{\sigma^3}(2x^2+1)\phi(x) + O(n^{-1})$$

$$\hat{G}_{T_n}(x) = P_*(T_n^* \leq x)$$

(can be shown) a.s. $= \Phi(x) + \frac{1}{\sqrt{n}} \frac{1}{6} \frac{\hat{\mu}_3}{\hat{\sigma}^3}(2x^2+1)\phi(x) + O(n^{-1})$

$\Rightarrow$

$$\sup_{x\in\mathbb{R}} \left| \hat{G}_{T_n}(x) - G_{T_n}(x) \right| = \underbrace{\frac{1}{\sqrt{n}} \, \frac{1}{6} \, \underbrace{\left| \frac{\hat{\mu}_3}{\hat{\sigma}^3} - \frac{\mu_3}{\sigma^3} \right|}_{O(n^{-1/2}) \text{ a.s.}}}_{O(n^{-1}) \text{ a.s.}} \sup_{x\in\mathbb{R}} \left| (2x^2+1)\phi(x) \right| + O(n^{-1})$$

faster convergence

Recall:

$$\sup_{x\in\mathbb{R}} \left| G_{T_n}(x) - \Phi(x) \right| = O(n^{-1/2})$$

Convergence is slower

Peter Hall.
The bootstrap and Edgeworth expansion.
Springer Science & Business Media, 2013.