Nested Designs & Random Effects

Timothy Hanson

Department of Statistics, University of South Carolina

Stat 506: Introduction to Design of Experiments

A production engineer studied the effects of machine model (factor A) and operator (factor B) on the output in a bottling plant. Three bottling machines were used, each a different model. Twelve different machine operators were studied, four on each machine working a six-hour shift each day for one week. The number of cases produced by each machine/operator are

Machine i:	1				2				3			
Operator j:	1	2	3	4	5	6	7	8	9	10	11	12
Day $k = 1$:	65	68	56	45	74	69	52	73	69	63	81	67
Day $k = 2$:	58	62	65	56	81	76	56	78	83	70	72	79
Day $k = 3$:	63	75	58	54	76	80	62	83	74	72	73	73
Day $k = 4$:	57	64	70	48	80	78	58	75	78	68	76	77
Day $k = 5$:	66	70	64	60	68	73	51	76	80	75	70	71

Each operator is only assigned *one* of the three treatments; operator is *nested* within treatment.

Model

The model is

$$y_{ijk} = \mu + \alpha_i + \beta_j + \epsilon_{ijk},$$

where $i = 1, 2, 3$ machines and $j = 1, \ldots, 12$ operators.

Note that this is different from a *crossed design* where every operator uses all three machines. Here, an operator is observed at one machine only.

"Nested" notation

We can also use different notation where the operator number starts at one *within each machine*. Then each pairing of (i, j) is a unique, *different* operator; this is commonly written as $j(i)$, operator j nested within machine i. Thus 2(1) (second operator at machine 1) is a *different person* than 2(2) (second operator at machine 2).

Machine i:			1				2				3	
Operator j:	1	2	3	4	1	2	3	4	1	2	3	4
Day $k = 1$:	65	68	56	45	74	69	52	73	69	63	81	67
Day $k = 2$:	58	62	65	56	81	76	56	78	83	70	72	79
Day $k = 3$:	63	75	58	54	76	80	62	83	74	72	73	73
Day $k = 4$:	57	64	70	48	80	78	58	75	78	68	76	77
Day $k = 5$:	66	70	64	60	68	73	51	76	80	75	70	71

Using this notation the model is

$$y_{ijk} = \mu + \alpha_i + \beta_{j(i)} + \epsilon_{ijk}.$$

Every operator has a unique $j(i)$.

ANOVA on simple nested design

Source	SS	df	MS	F
A	$SSA = bn \sum_{i=1}^{a} (\bar{y}_{i\bullet\bullet} - \bar{y}_{\bullet\bullet\bullet})^2$	$a-1$	MSA=SSA/dfA	MSA/MSE
B(A)	$SSB(A) = n \sum_{i=1}^{a} \sum_{j=1}^{b} (\bar{y}_{ij\bullet} - \bar{y}_{i\bullet\bullet})^2$	$a(b-1)$	SSB(A)/dfB(A)	MSB(A)/MSE
Error	$SSE = \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{n} (y_{ijk} - \bar{y}_{ij\bullet})^2$	$ab(n-1)$	SSE/dfE	

The F-tests are as usual and test $H_0 : \alpha_i = 0$ and $H_0 : \beta_{j(i)} = 0$ respectively.

```
library(cfcdae)
cases=c(65,68,56,45,74,69,52,73,69,63,81,67,
        58,62,65,56,81,76,56,78,83,70,72,79,
        63,75,58,54,76,80,62,83,74,72,73,73,
        57,64,70,48,80,78,58,75,78,68,76,77,
        66,70,64,60,68,73,51,76,80,75,70,71)
machine =factor(rep(c(1,1,1,1,2,2,2,2,3,3,3,3),5))
operator=factor(rep(c(1,2,3,4,1,2,3,4,1,2,3,4),5))
f=lm(cases~machine/operator) # note use of "/" to indicate nesting!
anova(f)
model.effects(f,machine)
pairwise(f,machine)
```

One-way random effects model

If treatment levels come from a larger population, their effects are best modeled as random. A random-effects one-way model is

$$y_{ij} = \mu + \alpha_i + \epsilon_{ij},$$

where

$$\alpha_1, \ldots, \alpha_g \stackrel{iid}{\sim} N(0, \sigma_\alpha^2) \text{ independent of } \epsilon_{ij} \stackrel{iid}{\sim} N(0, \sigma^2).$$

As usual, $i = 1, \ldots, g$ and $j = 1, \ldots, n_i$.

The test of interest is $H_0 : \alpha_1 = \cdots = \alpha_g = 0$. This happens if and only if $H_0 : \sigma_\alpha = 0$.

$\alpha_1, \ldots, \alpha_g$ are called *random effects* and σ_α and σ are termed *variance components*. This model is an example of a *random effects* model, because it has only random effects in it beyond the intercept μ (which is fixed).

Testing $H_0 : \sigma_\alpha = 0$

The MSE and MSTR are defined as they were before. One can show $E(MSE) = \sigma^2$ and $E(MSTR) = \sigma^2 + n\sigma_\alpha^2$ when $n = n_i$ for all i.

If $\sigma_\alpha = 0$ we expect $F^* = MSTR/MSE$ to estimate one. In fact, just like the fixed-effects case, $F^* \sim F(g - 1, N - g)$. This is the usual test given by `lm` followed by `anova`.

Apex Enterprises

$g = 5$ personnel officers were selected at random, and $n_i = 4$ prospective employee candidates assigned at random to each officer. y_{ij} is the rating of the ith officer on their jth candidate.

Since the personnel officers are chosen randomly from a large population of personnel officers, the random-effects one-way model applies, $y_{ij} = \mu + \alpha_i + \epsilon_{ij}$. exactRLRT provides a different test of $H_0 : \sigma_\alpha^2$ vs. $H_0 : \sigma_\alpha^2 > 0$.

```
rating=c(76,65,85,74,59,75,81,67,49,63,61,46,74,71,85,89,66,84,80,79)
officer=factor(c(1,1,1,1,2,2,2,2,3,3,3,3,4,4,4,4,5,5,5,5))
f=lm(rating~officer)
anova(f) # fixed effects model
AIC(f)

library(RLRsim)
f=lmer(rating~1+(1|officer),REML=F)
summary(f)
exactRLRT(f) # random effects model
```

Often the subsampled factor, e.g. operators, can be thought of as random,

$$\beta_{j(i)} \stackrel{iid}{\sim} N(0, \sigma_\beta^2).$$

However, it's easier to fit the nested model $y_{ijk} = \mu + \alpha_i + \beta_j + \epsilon_{ijk}$ in R where $\beta_j \stackrel{iid}{\sim} N(0, \sigma_\beta^2)$.

```
operator=factor(rep(1:12,5)) # each operator now has own index
f=lmer(cases~machine+(1|operator),REML=F)
summary(f)
exactRLRT(f) # are random operator effects significantly different?
library(lsmeans)
lsmeans(f,"machine") # machine effects averaged over operators
pairs(lsmeans(f,"machine")) # which machine(s) best?
library(car)
Anova(f,type=3)
```

Random block effects and repeated measures

When block levels come from a large population, we can consider a complete randomized block design with random block effects. One very important example of this is the repeated measures design, where each block is an experimental unit in which all treatment levels are randomly applied. In fact, the blocks are retermed "subjects" and we consider a sample of subjects from their population.

$$y_{ij} = \mu + \underbrace{\rho_i}_{\text{subject}} + \underbrace{\tau_j}_{\text{level}} + \epsilon_{ij},$$

where

$$\rho_1, \ldots, \rho_n \overset{iid}{\sim} N(0, \sigma_\rho^2) \text{ independent of } \epsilon_{ij} \overset{iid}{\sim} N(0, \sigma^2).$$

There are $i = 1, \ldots, n$ subjects receiving each of $j = 1, \ldots, r$ treatments.

Random block effects

Note that the model can be extended to factorial treatment structure, e.g.

$$Y_{ijk} = \mu + \rho_i + \alpha_j + \beta_k + (\alpha\beta)_{jk} + \epsilon_{ijk},$$

Examples of subjects include people, animals, families, cities, and clinics.

This is an example of a mixed effects model; there is a mix of random (ρ_i's) and fixed (α_j's, β_k's, and $(\alpha\beta)_{jk}$'s) effects in the model.

Random blocks, comments

- We assume subject effects and treatment effects do not interact. Can check via Tukey's 1 df test for additivity. Also look at interaction plots as in fixed-effects C.R.B. designs.
- ANOVA table, sums of squares exactly the same, except now the F-test for blocks tests $H_0 : \sigma_\rho = 0$ instead of $H_0 : \rho_1 = \cdots = \rho_n = 0$.
- Test for treatment is same $H_0 : \tau_1 = \cdots = \tau_r = 0$.
- Every treatment is given to every experimental unit in *randomized order*.
- Two sets of residuals to consider. Both should be normal; e_{ij} should have constant variance.
  1. $e_{ij} = y_{ij} - \{\hat{\mu} + \hat{\rho}_i + \hat{\tau}_j\}$, and
  2. $\hat{\rho}_i$.
- $\text{corr}(y_{ij_1}, y_{ij_2}) = \sigma_\rho^2 / (\sigma^2 + \sigma_\rho^2)$ for $j_1 \neq j_2$ tells you how correlated the repeated measures are.

A state highway department studied wear of five paints at eight randomly picked locations. The standard is paint 1. Paints 1, 3, and 5 are white; paints 2 and 4 are yellow. At each location a random ordering of the paints were applied to the road. After an exposure period, a combined measure of wear y_{ij} was recorded. The higher the score, the better the wearing characteristics.

Recall the model

$$y_{ij} = \mu + \underbrace{\rho_i}_{\text{location}} + \underbrace{\tau_j}_{\text{paint}} + \epsilon_{ij},$$

where

$$\rho_1, \ldots, \rho_n \overset{iid}{\sim} N(0, \sigma_\rho^2) \text{ independent of } \epsilon_{ij} \overset{iid}{\sim} N(0, \sigma^2).$$

Here $g = 5$ and $n = 8$.

```
location=factor(rep(1:8,each=5))
paint=factor(rep(1:5,times=8))
wear=c(11.0,13.0,10.0,18.0,15.0,20.0,28.0,15.0,30.0,18.0,
        8.0,10.0, 8.0,16.0,12.0,30.0,35.0,27.0,41.0,28.0,
       14.0,16.0,13.0,22.0,16.0,25.0,27.0,26.0,33.0,25.0,
       43.0,46.0,41.0,55.0,42.0,13.0,14.0,12.0,20.0,13.0)
d=data.frame(wear,location,paint)
with(d,interactplot(location,paint,wear)) # parallel?
source("http://people.stat.sc.edu/hansont/stat506/tukey.R")
tukeys.add.test(wear,location,paint)
f=lm(wear~paint+location)
anova(f)
AIC(f)

f=lmer(wear~paint+(1|location),REML=F)
summary(f)
exactRLRT(f) # are random location effects significantly different?
lsmeans(f,"paint") # paint effects
pairs(lsmeans(f,"paint")) # which paint(s) best?
# yellow vs. white:
contrast(lsmeans(f,"paint"),list(c1=c(-1/3,1/2,-1/3,1/2,-1/3)))
Anova(f,type=3) # whether paint is signicant
qqnorm((ranef(f)$location[,1])) # random effects normal?
plot(f) # different plot because fit w/ lmer
```

$r = 4$ Chardonnary wines of the same vintage were judged by $n = 6$ judges. Each wine was blinded and given to each judge in randomized order. The wines were scored on a 40-point scale y_{ij}, with higher scores meaning better wine.

The six judges are considered to come from a large population of wine-tasting judges and so a repeated measures model is appropriate.

Let's examine these data in R...

Wine tasting in R

```
rating=c(20,24,28,28,15,18,23,24,18,19,24,23,26,26,30,30,22,24,28,26,19,21,27,25)
judge=factor(c(1,1,1,1,2,2,2,2,3,3,3,3,4,4,4,4,5,5,5,5,6,6,6,6))
wine=factor(c(1,2,3,4,1,2,3,4,1,2,3,4,1,2,3,4,1,2,3,4,1,2,3,4))
d=data.frame(rating,judge,wine)
with(d,interactplot(judge,wine,rating)) # parallel?
tukeys.add.test(rating,wine,judge)
```