

Appendix B

Common Probability Distributions

This appendix serves as a reminder of the parametric forms used in the text. Considerably more detail can be found in the standard references: Johnson et al. (2005) for univariate forms on the counting measure; Johnson, Kotz, and Balakrishnan (1997) for multivariate forms on the counting measure; Johnson, Kotz, and Balakrishnan (1994, 1995) for univariate forms on the Lebesgue measure; Johnson, Kotz, and Balakrishnan (2000) for multivariate forms on the Lebesgue measure; Fang, Kotz, and Ng (1990) concentrating on symmetric forms; Kotz and Nadarajah (2000) for extreme value distributions; and more generally Evans, Hastings, and Peacock (2000), Balakrishnan and Nevzorov (2003), and Krishnamoorthy (2006).

- **Bernoulli**

- PMF: $\mathcal{BR}(x|p) = p^x(1-p)^{1-x}, x = 0, 1, \quad 0 < p < 1.$
- $E[X] = p.$
- $Var[X] = p(1-p).$

- **Beta**

- PDF: $\mathcal{BE}(x|\alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}, \quad 0 < x < 1, 0 < \alpha, \beta.$
- $E[X] = \frac{\alpha}{\alpha+\beta}.$
- $Var[X] = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}.$

- **Binomial**

- PMF: $\mathcal{BN}(x|n, p) = \binom{n}{x}p^x(1-p)^{n-x}, x = 0, 1, \dots, n, \quad 0 < p < 1.$
- $E[X] = np.$
- $Var[X] = np(1-p).$

- **Cauchy**

- PDF: $\mathcal{C}(x|\theta, \sigma) = \frac{1}{\pi\sigma} \frac{1}{1 + \left(\frac{x-\theta}{\sigma}\right)^2}, \quad -\infty < x, \theta < \infty, 0 < \sigma.$

- $E[X]$ = Does not exist.
- $Var[X]$ = Does not exist.

• **Dirichlet**

- PDF: $\mathcal{D}(\mathbf{x}|\alpha_1, \dots, \alpha_k) = \frac{\Gamma(\alpha_1 + \dots + \alpha_k)}{\Gamma(\alpha_1) \dots \Gamma(\alpha_k)} x_1^{\alpha_1 - 1} \dots x_k^{\alpha_k - 1} \quad 0 \leq x_i \leq 1, \sum_{i=1}^k x_i = 1, 0 < \alpha_i, \forall i \in [1, 2, \dots, k]$.
- $E[X_i] = \frac{\alpha_i}{\alpha_0}$, where $\alpha_0 = \sum_{j=1}^k \alpha_j$.
- $Var[X_i] = \frac{\alpha_i(\alpha_0 - \alpha_i)}{\alpha_0^2(\alpha_0 + 1)}$.
- $Cov[X_h, X_i] = -\frac{\alpha_h \alpha_i}{\alpha_0^2(\alpha_0 + 1)}$.

• **Double Exponential**

- PDF: $\mathcal{DE}(x|\mu, \dots, \tau) = \frac{1}{2\tau} \exp[-|x - \mu|/\tau] \quad -\infty < \mu, x < \infty, 0 < \tau$.
- $E[X] = \mu$.
- $Var[X] = 2\tau^2$.

• **F**

- PDF: $\mathcal{F}(x|\nu_1, \nu_2) = \frac{\Gamma(\frac{\nu_1 + \nu_2}{2})}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} \left(\frac{\nu_1}{\nu_2}\right)^{\nu_1/2} \frac{x^{\nu_1 - 2/2}}{(1 + \frac{\nu_1}{\nu_2}x)^{\frac{\nu_1 + \nu_2}{2}}}$,
 $0 \leq x < \infty, \nu_1, \nu_2 \in \mathbb{I}^+$.
- $E[X] = \frac{\nu_2}{\nu_2 - 2}, \nu_2 > 2$.
- $Var[X] = 2 \left(\frac{\nu_2}{\nu_2 - 2}\right)^2 \frac{\nu_1 + \nu_2 - 2}{\nu_1(\nu_2 - 4)}, \nu_2 > 4$.

• **Gamma**

- PDF: $\mathcal{G}(x|\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha - 1} \exp[-x\beta], \quad 0 \leq x < \infty, \quad 0 < \alpha, \beta$.
- $E[X] = \frac{\alpha}{\beta}$.
- $Var[X] = \frac{\alpha}{\beta^2}$.
- Note: the χ^2 distribution is $\mathcal{G}(\frac{df}{2}, \frac{1}{2})$, and the exponential distribution ($\mathcal{EX}(\beta)$) is $\mathcal{G}(1, \beta)$.

• **Geometric**

- PMF: $\mathcal{GEO}(x|p) = p(1 - p)^{x-1}, \quad x = 1, 2, \dots, \quad 0 \leq p \leq 1$.
- $E[X] = \frac{1}{p}$.
- $Var[X] = \frac{1-p}{p^2}$.

• **Hypergeometric**

- PMF: $\mathcal{HG}(x|n, m, k) = \frac{\binom{m}{x} \binom{n-m}{k-x}}{\binom{n}{k}}, \quad m - n + k \leq x \leq m, \quad n, m, k \geq 0$.

$$- E[X] = \frac{km}{n}.$$

$$- Var[X] = \frac{km(n-m)(n-k)}{n^2(n-1)}.$$

- **Inverse Gamma**

$$- \text{PDF: } \mathcal{IG}(x|\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} (x)^{-(\alpha+1)} \exp[-\beta/x], \quad 0 < x, \alpha, \beta.$$

$$- E[X] = \frac{\beta}{\alpha-1}, \alpha > 1.$$

$$- Var[X] = \frac{\beta^2}{(\alpha-1)^2(\alpha-2)}, \alpha > 2.$$

- **Lognormal**

$$- \text{PDF: } \mathcal{LN}(x|\mu, \sigma) = (2\pi\sigma^2)^{-\frac{1}{2}} x^{-1} \exp[-(\log(x) - \mu)^2/2\sigma^2],$$

$$-\infty < \mu, x < \infty, 0 < \sigma^2$$

$$- E[X] = \exp[\mu + \sigma^2/2]$$

$$- Var[X] = \exp[2(\mu + \sigma^2)] - \exp[2\mu + \sigma^2].$$

- **Multinomial**

$$- \text{PMF: } \mathcal{MN}(x|n, p_1, \dots, p_k) = \frac{n!}{x_1! \dots x_k!} p_1^{x_1} \dots p_k^{x_k}, \quad x_i = 0, 1, \dots, n,$$

$$0 < p_i < 1, \quad \sum_{i=1}^k p_i = 1.$$

$$- E[X_i] = np_i.$$

$$- Var[X_i] = np_i(1 - p_i).$$

$$- Cov[X_i, X_j] = -np_i p_j.$$

- **Negative Binomial**

$$- \text{PMF: } \mathcal{NB}(x|r, p) = \binom{x-1}{r-1} p^r (1-p)^{x-r}, \quad x = 0, 1, \dots, \quad 0 < p < 1,$$

$$r \in \mathcal{I}^+.$$

$$- E[X] = \frac{r(1-p)}{p}.$$

$$- Var[X] = \frac{r(1-p)}{p^2}.$$

- **Normal**

$$- \text{PDF: } \mathcal{N}(x|\mu, \sigma^2) = (2\pi\sigma^2)^{-\frac{1}{2}} \exp\left[-\frac{1}{2\sigma^2}(x - \mu)^2\right],$$

$$-\infty < \mu, x < \infty, 0 < \sigma.$$

$$- E[X] = \mu.$$

$$- Var[X] = \sigma^2.$$

$$\text{Multivariate case: } \mathcal{N}_k(\mathbf{x}|\boldsymbol{\mu}, \boldsymbol{\Sigma}^2) = (2\pi)^{-k/2} |\boldsymbol{\Sigma}|^{-1/2} \exp\left[-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})' \boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu})\right]$$

- **Pareto**

$$- \text{PDF: } \mathcal{PA}(x|\alpha, \beta) = \alpha\beta^\alpha x^{-(\alpha+1)}, \quad \beta < x, 0 < \alpha, \beta.$$

- $E[X] = \frac{\beta\alpha}{\alpha-1}$, exists provided $\alpha > 1$.
- $Var[X] = \frac{\beta^2\alpha}{(\alpha-1)^2(\alpha-2)}$, exists provided $\alpha > 2$.

- **Poisson**

- PMF: $\mathcal{P}(x|\lambda) = \frac{\lambda^x e^{-\lambda}}{x!}$, $x = 0, 1, \dots$, $0 \leq \lambda < \infty$.
- $E[X] = \lambda$.
- $Var[X] = \lambda$.

- **t**

- PDF: $\mathcal{T}(x|\theta) = \frac{\Gamma(\frac{\theta+1}{2})}{\Gamma(\frac{\theta}{2})} \frac{1}{(\pi\theta)^{\frac{1}{2}} (1+x^2/\theta)^{(\theta+1)/2}}$, $-\infty < \mu, x < \infty, \theta \in \mathbb{I}^+$.
- $E[X] = 0, 1 < \theta$.
- $Var[X] = \frac{\theta}{\theta-2}, 2 < \theta$.

- **t, Multivariate**

- PDF: $\mathcal{MVT}(\mathbf{x}|\mathbf{M}, \theta) = |\mathbf{M}|^{-\frac{1}{2}} (\pi\theta)^{-k/2} \frac{\Gamma(\frac{\theta+k}{2})}{\Gamma(\frac{\theta}{2})} \left(1 + \frac{(\mathbf{x}-\boldsymbol{\mu})'\mathbf{M}(\mathbf{x}-\boldsymbol{\mu})}{\theta}\right)^{-\frac{\theta+k}{2}}$,
where \mathbf{x} is a k -length vector, \mathbf{M} is a $k \times k$ positive definite matrix, and θ is a positive scalar.
- $E[\mathbf{X}] = \boldsymbol{\mu}$.
- $Var[\mathbf{X}] = \frac{\theta}{\theta-2} \mathbf{M}^{-1}$.

- **Uniform**

k -Category Discrete Case PMF:

- $\mathcal{U}(x) = p(X = x) = \begin{cases} \frac{1}{k}, & \text{for } x = 1, 2, \dots, k \\ 0, & \text{otherwise} \end{cases}$
- $E[X] = \frac{k+1}{2}$.
- $Var[X] = \frac{(k+1)(k-1)}{12}$.

Continuous Case PDF:

- $\mathcal{U}(x) = f(x) = \begin{cases} \frac{1}{b-a}, & \text{for } a = 0 \leq x \leq b = 1 \\ 0, & \text{otherwise} \end{cases}$
- $E[X] = \frac{b-a}{2}$.
- $Var[X] = \frac{(b-a)^2}{12}$.

- **Wishart**

- PDF: $\mathcal{W}(\mathbf{X}|\alpha, \boldsymbol{\beta}) = \frac{|\mathbf{X}|^{(\alpha-(k+1))/2}}{\Gamma_k(\alpha)|\boldsymbol{\beta}|^{\alpha/2}} \exp[-\text{tr}(\boldsymbol{\beta}^{-1}\mathbf{X})/2]$
where: $\Gamma_k(\alpha) = 2^{\alpha k/2} \pi^{k(k-1)/4} \prod_{i=1}^k \Gamma\left(\frac{\alpha+1-i}{2}\right)$, $2\alpha > k-1$,
 $\boldsymbol{\beta}$ symmetric nonsingular, and \mathbf{X} symmetric positive definite.