Appendix B

Common Probability Distributions

This appendix serves as a reminder of the parametric forms used in the text. Considerably more detail can be found in the standard references: Johnson et al. (2005) for univariate forms on the counting measure; Johnson, Kotz, and Balakrishnan (1997) for multivariate forms on the counting measure; Johnson, Kotz, and Balakrishnan (1994, 1995) for univariate forms on the Lebesgue measure; Johnson, Kotz, and Balakrishnan (2000) for multivariate forms on the Lesbegue measure, Fang, Kotz, and Ng (1990) concentrating on symmetric forms; Kotz and Nadarajah (2000) for extreme value distributions; and more generally Evans, Hastings, and Peacock (2000), Balakrishnan and Nevzorov (2003), and Krishnamoorthy (2006).

- **Bernoulli**
  - PMF: $\mathcal{BR}(x|p) = p^x(1-p)^{1-x}, x = 0, 1, \quad 0 < p < 1.$
  - $E[X] = p.$
  - $Var[X] = p(1-p).$
- **Beta**
  - PDF: $\mathcal{BE}(x|\alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \quad 0 < x < 1, 0 < \alpha, \beta.$
  - $E[X] = \frac{\alpha}{\alpha+\beta}.$
  - $Var[X] = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}.$
- **Binomial**
  - PMF: $\mathcal{BN}(x|n, p) = \binom{n}{x} p^x(1-p)^{n-x}, x = 0, 1, \ldots, n, \quad 0 < p < 1.$
  - $E[X] = np.$
  - $Var[X] = np(1-p).$
- **Cauchy**
  - PDF: $\mathcal{C}(x|\theta, \sigma) = \frac{1}{\pi\sigma} \frac{1}{1+(\frac{x-\theta}{\sigma})^2}, \quad -\infty < x, \theta < \infty, 0 < \sigma.$

- $E[X] =$ Does not exist.
- $Var[X] =$ Does not exist.

- **Dirichlet**
  - PDF: $\mathcal{D}(\mathbf{x}|\alpha_1,\ldots,\alpha_k) = \frac{\Gamma(\alpha_1+\ldots+\alpha_k)}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_k)} x_1^{\alpha_1-1}\cdots x_k^{\alpha_k-1} \quad 0 \le x_i \le 1, \sum_{i=1}^k x_i = 1, 0 < \alpha_i, \forall i \in [1,2,\ldots,k]$.
  - $E[X_i] = \frac{\alpha_i}{\alpha_0}$, where $\alpha_0 = \sum_{j=1}^k \alpha_j$.
  - $Var[X_i] = \frac{\alpha_i(\alpha_0-\alpha_i)}{\alpha_0^2(\alpha_0+1)}$.
  - $\text{Cov}[X_h, X_i] = -\frac{\alpha_h\alpha_i}{\alpha_0^2(\alpha_0+1)}$.

- **Double Exponential**
  - PDF: $\mathcal{DE}(x|\mu,\ldots,\tau) = \frac{1}{2\tau}\exp[-|x-\mu|/\tau] \quad -\infty < \mu, x < \infty, 0 < \tau$.
  - $E[X] = \mu$.
  - $Var[X] = 2\tau^2$.

- **F**
  - PDF: $\mathcal{F}(x|\nu_1,\nu_2) = \frac{\Gamma\left(\frac{\nu_1+\nu_2}{2}\right)}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})}\left(\frac{\nu_1}{\nu_2}\right)^{\nu_1/2} \frac{x^{(\nu_1-2)/2}}{\left(1+\frac{\nu_1}{\nu_2}x\right)^{\frac{\nu_1+\nu_2}{2}}}$,
    $0 \le x < \infty, \nu_1, \nu_2 \in \mathbb{I}^+$.
  - $E[X] = \frac{\nu_2}{\nu_2-2}, \nu_2 > 2$.
  - $Var[X] = 2\left(\frac{\nu_2}{\nu_2-2}\right)^2 \frac{\nu_1+\nu_2-2}{\nu_1(\nu_2-4)}, \quad \nu_2 > 4$.

- **Gamma**
  - PDF: $\mathcal{G}(x|\alpha,\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1}\exp[-x\beta], \quad 0 \le x < \infty, \quad 0 < \alpha, \beta$.
  - $E[X] = \frac{\alpha}{\beta}$.
  - $Var[X] = \frac{\alpha}{\beta^2}$.
  - Note: the χ^2 distribution is $\mathcal{G}\left(\frac{df}{2}, \frac{1}{2}\right)$, and the exponential distribution ($\mathcal{EX}(\beta)$) is $\mathcal{G}(1,\beta)$.

- **Geometric**
  - PMF: $\mathcal{GEO}(x|p) = p(1-p)^{x-1}, \quad x = 1,2,\ldots, \quad 0 \le p \le 1$.
  - $E[X] = \frac{1}{p}$.
  - $Var[X] = \frac{1-p}{p^2}$.

- **Hypergeometric**
  - PMF: $\mathcal{HG}(x|n,m,k) = \frac{\binom{m}{x}\binom{n-m}{k-x}}{\binom{n}{k}}, \quad m-n+k \le x \le m, \quad n,m,k \ge 0$.

- $E[X] = \frac{km}{n}$.
- $Var[X] = \frac{km(n-m)(n-k)}{n^2(n-1)}$.

- **Inverse Gamma**
  - PDF: $\mathcal{IG}(x|\alpha,\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)}(x)^{-(\alpha+1)}\exp[-\beta/x], \quad 0 < x, \alpha, \beta$.
  - $E[X] = \frac{\beta}{\alpha-1}, \alpha > 1$.
  - $Var[X] = \frac{\beta^2}{(\alpha-1)^2(\alpha-2)}, \alpha > 2$.

- **Lognormal**
  - PDF: $\mathcal{LN}(x|\mu,\sigma) = (2\pi\sigma^2)^{-\frac{1}{2}}x^{-1}\exp[-(\log(x)-\mu)^2/2\sigma^2]$, $-\infty < \mu, x < \infty, 0 < \sigma^2$
  - $E[X] = \exp[\mu + \sigma^2/2]$
  - $Var[X] = \exp[2(\mu+\sigma^2)] - \exp[2\mu+\sigma^2]$.

- **Multinomial**
  - PMF: $\mathcal{MN}(x|n,p_1,\ldots,p_k) = \frac{n!}{x_1!\cdots x_k!}p_1^{x_1}\cdots p_k^{x_k}, \quad x_i = 0,1,\ldots,n$, $0 < p_i < 1, \quad \sum_{i=1}^k p_i = 1$.
  - $E[X_i] = np_i$.
  - $Var[X_i] = np_i(1-p_i)$.
  - $\text{Cov}[X_i, X_j] = -np_ip_j$.

- **Negative Binomial**
  - PMF: $\mathcal{NB}(x|r,p) = \binom{x-1}{r-1}p^r(1-p)^{x-r}, \quad x = 0,1,\ldots, \quad 0 < p < 1$, $r \in \mathcal{I}^+$.
  - $E[X] = \frac{r(1-p)}{p}$.
  - $Var[X] = \frac{r(1-p)}{p^2}$.

- **Normal**
  - PDF: $\mathcal{N}(x|\mu,\sigma^2) = (2\pi\sigma^2)^{-\frac{1}{2}}\exp\left[-\frac{1}{2\sigma^2}(x-\mu)^2\right]$, $-\infty < \mu, x < \infty, 0 < \sigma$.
  - $E[X] = \mu$.
  - $Var[X] = \sigma^2$.

  Multivariate case: $\mathcal{N}_k(\mathbf{x}|\boldsymbol{\mu},\boldsymbol{\Sigma}^2) = (2\pi)^{-k/2}|\boldsymbol{\Sigma}|^{-1/2}\exp\left[-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu}\right.$

- **Pareto**
  - PDF: $\mathcal{PA}(x|\alpha,\beta) = \alpha\beta^\alpha x^{-(\alpha+1)}, \quad \beta < x, 0 < \alpha, \beta$.

- $E[X] = \frac{\beta\alpha}{\alpha-1}$, exists provided $\alpha > 1$.
- $Var[X] = \frac{\beta^2\alpha}{(\alpha-1)^2(\alpha-2)}$, exists provided $\alpha > 2$.

- **Poisson**
  - PMF: $\mathcal{P}(x|\lambda) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0, 1, \ldots, \quad 0 \le \lambda < \infty.$
  - $E[X] = \lambda$.
  - $Var[X] = \lambda$.

- **t**
  - PDF: $\mathcal{T}(x|\theta) = \frac{\Gamma\left(\frac{\theta+1}{2}\right)}{\Gamma\left(\frac{\theta}{2}\right)} \frac{1}{(\pi\theta)^{\frac{1}{2}}(1+x^2/\theta)^{(\theta+1)/2}}, \quad -\infty < \mu, x < \infty, \theta \in \mathbb{I}^+.$
  - $E[X] = 0, 1 < \theta$.
  - $Var[X] = \frac{\theta}{\theta-2}, 2 < \theta$.

- **t, Multivariate**
  - PDF: $\mathcal{MVT}(\mathbf{x}|\mathbf{M}, \theta) = |\mathbf{M}|^{-\frac{1}{2}}(\pi\theta)^{-k/2}\frac{\Gamma\left(\frac{\theta+k}{2}\right)}{\Gamma\left(\frac{\theta}{2}\right)}\left(1 + \frac{(\mathbf{x}-\boldsymbol{\mu})'\mathbf{M}(\mathbf{x}-\boldsymbol{\mu})}{\theta}\right)^{-\frac{\theta+k}{2}}$, where $\mathbf{x}$ is a k-length vector, $\mathbf{M}$ is a $k \times k$ positive definite matrix, and θ is a positive scalar.
  - $E[\mathbf{X}] = \boldsymbol{\mu}$.
  - $Var[\mathbf{X}] = \frac{\theta}{\theta-2}\mathbf{M}^{-1}$.

- **Uniform**

  k-Category Discrete Case PMF:
  - $\mathcal{U}(x) = p(X = x) = \begin{cases} \frac{1}{k}, & \text{for } x = 1, 2, \ldots, k \\ 0, & \text{otherwise} \end{cases}$
  - $E[X] = \frac{k+1}{2}$.
  - $Var[X] = \frac{(k+1)(k-1)}{12}$.

  Continuous Case PDF:
  - $\mathcal{U}(x) = f(x) = \begin{cases} \frac{1}{b-a}, & \text{for } a = 0 \le x \le b = 1 \\ 0, & \text{otherwise} \end{cases}$
  - $E[X] = \frac{b-a}{2}$.
  - $Var[X] = \frac{(b-a)^2}{12}$.

- **Wishart**
  - PDF: $\mathcal{W}(\mathbf{X}|\alpha, \boldsymbol{\beta}) = \frac{|\mathbf{X}|^{(\alpha-(k+1))/2}}{\Gamma_k(\alpha)|\boldsymbol{\beta}|^{\alpha/2}} \exp[-\text{tr}(\boldsymbol{\beta}^{-1}\mathbf{X})/2]$ where: $\Gamma_k(\alpha) = 2^{\alpha k/2}\pi^{k(k-1)/4}\prod_{i=1}^{k}\Gamma\left(\frac{\alpha+1-i}{2}\right)$, $2\alpha > k-1$, $\boldsymbol{\beta}$ symmetric nonsingular, and $\mathbf{X}$ symmetric positive definite.