STAT 521 – Spring 2019 – HW 2

Note: You must show your work wherever possible!

1. A Markov chain $\{X_n, n \geq 0\}$ with states 0, 1, 2, has the transition probability matrix

$$\begin{bmatrix} 1/2 & 1/3 & 1/6 \\ 0 & 1/3 & 2/3 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

If $P[X_0 = 0] = P[X_0 = 1] = 1/4$, then find $E(X_3)$ [Hint: Note that you must work with the *unconditional* distribution of X_3].

2. On any given day, I am either cheerful (C), so-so (S), or glum (G). If I am cheerful today, then I will be C, S, or G tomorrow with respective probabilities 0.5, 0.4, 0.1. If I am so-so today, then I will be C, S, or G tomorrow with probabilities 0.3, 0.4, 0.3. If I am glum today, then I will be C, S, or G tomorrow with probabilities 0.2, 0.3, 0.5. I am currently in a cheerful mood. What is the probability that I am not in a glum mood on any of the following three days? [Hint: Create a chain with an absorbing state.]

3. Identify the classes of the following Markov chains having the transition probability matrices below, and determine whether they are transient or recurrent:

(a)

$$P_1 = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

(b)

$$P_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

(c)

$$P_3 = \begin{bmatrix} 1/2 & 0 & 1/2 & 0 & 0 \\ 1/4 & 1/2 & 1/4 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 1/2 & 1/2 \end{bmatrix}$$

(d)

$$P_4 = \begin{bmatrix} 1/4 & 3/4 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1/3 & 2/3 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

4. Coin 1 comes up heads with probability 0.6 and coin 2 comes up heads with probability 0.5. In an experiment, one of these coins is continually flipped until it comes up tails, at which time that coin is put aside and we start flipping the other one.
(a) In the long run, what proportion of flips use coin 1?
(b) If we start the process with coin 1, what is the probability that coin 2 is used on the fifth flip?

5. In a good weather year, the number of storms is Poisson distributed with mean 1; in a bad year, it is Poisson distributed with mean 3. Suppose that any year's weather type depends on past years ONLY through the previous year's type. Suppose a good year is equally likely to be followed by either a good year or a bad year, and that a bad year is twice as likely to be followed by a bad year as by a good year. Suppose that last year (call it year 0) was a good year. Let the process $\{X_n\}$ represent the weather (state 0 = good, state 1 = bad) in year n. Let T_i = the number of storms in year i.
(a) Find the expected total number of storms in the next two years (that is, in years 1 and 2). [Hint: Condition on the weather types for years 1 and 2.]
(b) Find the probability that there are no storms in year 3. [Hint: Condition on the weather type for year 3.]
(c) Find the long-run average number of storms per year. [Hint: First find the long-run proportion of years for each weather type.]

6. Three out of every four trucks on the road are followed by a car, while only one out of every five cars on the road is followed by a truck. In the long run, what proportion of vehicles on the road are trucks?

Graduate Students Must Also Do The Following Problem (Undergraduates may do it for extra credit):

7. The state of a process changes daily according to a two-state Markov chain. If the process is in state i during one day, then it is in state j the following day with probability P_{ij}, where
$$P_{00} = 0.4, P_{01} = 0.6, P_{10} = 0.2, P_{11} = 0.8.$$
Every day a message is sent. If the state of the Markov chain that day is i (where i is either 0 or 1), then the message sent is "good" with probability p_i and is "bad" with probability $q_i = 1 - p_i$. [Hint: Solve (a) and (b) below by conditioning!]

(a) If the process is in state 0 on Monday, what is the probability that a good message is sent on Tuesday?

(b) If the process is in state 0 on Monday, what is the probability that a good message is sent on Friday?

(c) In the long run, what proportion of messages are good?