STAT 521 – Spring 2019 – HW 5

Note: You must show your work wherever possible! Problems 1(a) and 2 through 5 are required for everyone; problem 1(b,c,d) is required for graduate students and optional (extra credit) for undergraduates.

1. Suppose two individuals, Alice and Barry, both require kidney transplants. Suppose that if she does not receive a new kidney, Alice will die after a length of time that is exponential with rate μ_A. Also, if he does not receive a new kidney, Barry will die after a length of time that is exponential with rate μ_B. New kidneys arrive according to a Poisson process having rate λ. It has been decided the the first kidney to arrive will go to Alice (or to Barry if Barry is alive and Alice is not at that time) and the next one will go to Barry (if he is still living).

   [Note: Part (a) is required for everyone. Parts (b), (c), (d) are required for graduate students and optional (extra credit) for undergraduates.]

   (a) What is the probability that Alice obtains a new kidney?

   (b) What is the probability that Barry obtains a new kidney?

   (c) What is the probability that neither Alice nor Barry obtains a new kidney?

   (d) What is the probability that both Alice and Barry receive new kidneys?

2. Let $\{N(t)\}$ be a Poisson process with rate λ. Let S_n denote the time of the n-th event.

   (a) Find $E(S_4)$.

   (b) Find $E(S_4|N(1) = 2)$.

   (c) Find $E[N(4) - N(2)|N(1) = 3]$.

3. Suppose that events occur according to a Poisson process with rate λ. Each time an event occurs, we must decide whether or not to stop, with our objective being to stop at the last event to occur prior to some specified time T, where $T > 1/\lambda$. That is, if an event occurs at time t, (where $0 \leq t \leq T$), and we decide to stop, then we win if there are no additional events by time T, and we lose otherwise. If we do not stop when an event occurs and no additional events occur by time T, then we lose. Also, if no events occur at all by time T, then we lose. Consider the strategy that stops at the first event to occur after some fixed time s, where $0 \leq s \leq T$.

   (a) Using this strategy, what is the probability of winning? (Note the answer will be a function of s and T.) [Hint: Write the probability of winning as a probability that involves $N(T)$ and $N(s)$.]
   (b) What value of s maximizes the probability of winning?
   (c) Show that one's probability of winning when using the preceding strategy with the value of s specified in part (b) is $1/e$.

4. The number of hours between successive train arrivals at the station is uniformly distributed on the interval $(0, 1)$. Passengers for the train arrive according to a Poisson process with rate 7 per hour. Suppose a train has just left the station. Let X denote the number of people who get on the next train. [Hint: You can solve the following by either conditioning on the uniform random variable, or by directly applying the "iterated" formulas.]

   (a) Find $E[X]$.
   (b) Find $var[X]$.

5. Suppose there are two types of claims that are made to an insurance company. Let $N_i(t)$ denote the number of Type i claims made by time t, and suppose that $\{N_1(t), t \geq 0\}$ and $\{N_2(t), t \geq 0\}$ are independent Poisson processes with respective rates $\lambda_1 = 10$ and $\lambda_2 = 1$. The dollar amounts of Type 1 claims are independent exponential random variables with mean \$1000 whereas the dollar amounts of Type 2 claims are independent exponential random variables with mean \$5000. A claim for \$4000 has just been received. What is the probability that it is a Type 1 claim? **NOTE:** You may assume that only whole-dollar claims are possible, so that the probability that a claim is, say, \$D can be approximated by plugging D into the appropriate pdf (even though the pdf is continuous). [HINT: Find $p =$ P[a claim is Type 1], and then use Bayes' formula.]