STAT 521 – Spring 2019 – HW 7

Note: You must show your work wherever possible! Problems 1(a,b) and 2, 4, 5, 7, 8 are required for everyone; problems 1(c), 3, 6 are required for graduate students and optional (extra credit) for undergraduates.

1. A small barbershop, operated by a single barber, has room for at most two customers. Potential customers arrive according to a Poisson process with rate 3 per hour. The successive service times are independent exponential random variables with mean 0.25 hours. Let $X(t)$ be the number of customers in the shop at time t.

   [Note: Parts (a) and (b) are required for everyone. Part (c) is required for graduate students and optional (extra credit) for undergraduates.]

   (a) What is the long-run average number of customers in the shop? [Hint: Find the long-run proportions of time the chain spends in each state.]

   (b) Of all the store's **potential** customers, what proportion actually enter the shop?

   (c) **Graduate students:** Explain how the answer to part (b) would change if the barber's service rate were twice as fast.

2. Potential customers arrive at a full-service, one-pump gas station according to a Poisson process with a rate of 20 cars per hour. However, customers will only enter the station for gas if there are two or fewer cars (including the one currently being attended to) at the pump. Suppose the amount of time required to service a car is exponential with mean five **minutes**. Let $X(t)$ be the number of cars in the system at time t.

   (a) In the long run, what fraction of the attendant's time is spent servicing cars? [Hint: Find the long-run proportions of time the chain spends in each state. During which state(s) is the attendant servicing cars?]

   (b) Of all the station's **potential** customers, what proportion drive off without entering the station?

3. **Graduate student problem:** After being repaired, a machine functions for an exponential time with rate λ and then fails. Upon a failure, a repair process begins. The repair process proceeds sequentially through k distinct phases. First a phase 1 repair must be performed, then a phase 2, and so on. The times to complete these phases are independent, with phase i taking an exponential time with rate μ_i, $i = 1, \ldots, k$.

   (a) What proportion of time is the machine undergoing a phase i repair?
   (b) What proportion of time is the machine working?

4. A job shop consists of 3 machines and two repairmen. The amount of time a machine works before breaking down is exponentially distributed with mean 10. Suppose the amount of time it takes a single repairman to fix a machine is exponentially distributed with mean 8. Let $X(t)$ be the number of **broken** machines at time t.

(a) In the long run, what is the average number of machines not in use? [Hint: Find the long-run proportions of time the chain spends in each state.]
(b) In the long run, what proportion of time are both repairmen busy?

5. Recall the situation of toddler Todd from Homework 6. Let $X(t)$ be Todd's status at time t, and label his status at any given time as follows: 0=asleep, 1=happy, 2=sad, 3=angry.

   (a) Set up the balance equations for the process $\{X(t)\}$ describing Todd's status.
   (b) In the long run, what proportion of time is Todd angry?
   (c) If Todd is currently asleep, then using R, approximate the probability that 24 hours from now, Todd will be happy.

6. **Graduate student problem:** Let $\{B(t)\}$ be a standard Brownian motion process. Give the conditional distribution of $B(s)$ given the $B(t_1) = A$ and $B(t_2) = B$, where $0 < t_1 < s < t_2$.

7. Prove that $P[T_a < \infty] = 1$, where T_a is the time it takes for a standard Brownian motion process to hit value a.

8. Suppose you own one share of a stock whose price changes according to a standard Brownian motion process. Suppose that you purchased the stock at a price $b+c$, $c > 0$, and the present price is b. You have decided to sell that stock either when it reaches the price $b + c$ or when an additional time t goes by (whichever occurs first). Give an expression for the probability that you do not recover your original purchase price of $b + c$. [HINT: Consider the process $\{X(t)\}$, where $X(t)$ is the change in price (up or down from the current price of b) at a time t time units after the present time. Note what $X(0)$ equals, and note what type of process is $\{X(t)\}$.]