STAT 509 – Sections 7.1-7.2: Design of Experiments

• An engineer conducted an experiment to develop a prediction equation for the distance a catapult can throw a metal ball.
• He manipulated two factors: (A) The angle of the catapult arm (measured in degrees) and (B) the height of the pin that supports the rubber band, measured at equally spaced locations.
• For each factor, two levels were examined, a low level and a high level.
• Angle Levels:
Height Levels:
• In the data set, we will always code the low level of a factor as –1 and code the high level as 1.
• Two distance measurements were taken at each <u>treatment</u> (factor level combination):

Angle	Height	Distance 1	Distance 2
–1	–1	27	27
1	–1	81	67
–1	1	67	62
1	1	137	158

• A 2^k factorial design measures a response at each treatment, using *k* factors each having 2 levels.

• The above experiment is a factorial design.
Picture:

<u>Notation for Mean Responses in a 2^2 Factorial Design</u>

- **a = Average of responses when A (Angle) is high and B (Height) is low.**
- **b = Average of responses when B (Height) is high and A (Angle) is low.**
- **ab = average of responses when both A (Angle) and B (Height) are high.**
- **(1) = Average of responses when both A (Angle) and B (Height) are low.**

Catapult Experiment:

- **The main effect of a factor in a 2^k factorial experiment is: (mean response at the high level of the factor) – (mean response at the low level of the factor)**

Estimated main effect of Angle:

Estimated main effect of Height:

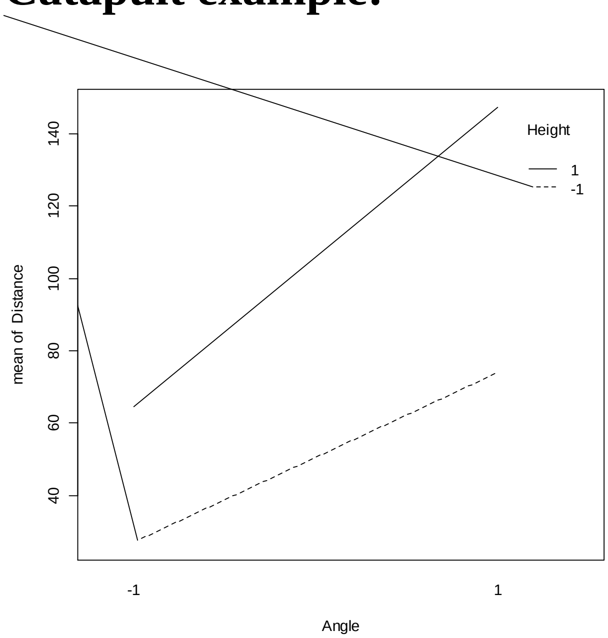
<u>Interactions</u>

- **Sometimes there is interaction between two factors.**

- **If Angle and Height interact, then the effect of Angle on the response *differs*, depending on the level of Height (and similarly, the effect of Height on the response depends on which level of Angle we are at).**

- **Whether or not two factors interact can be indicated by an <u>interaction plot</u>.**
- **An interaction plot plots the means of one factor, *separately at each level of the other factor*.**

Catapult example:



- **If the lines in an interaction plot are (nearly) parallel, this indicates NO interaction.**

- **If the lines in an interaction plot are non-parallel, this indicates interaction between the factors.**

Linear Model for the 2^2 Factorial Design

$$Y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_{12} x_{i1} x_{i2} + \varepsilon_i$$

• A slope represents the expected change in the response when we increase one factor by one unit while holding the other factor constant.
• Going from -1 to 1 in a factor represents movement of two units.
So:

• In practice, software can produce these estimates:

R code:

```
> y <- c(27, 27, 81, 67, 67, 62, 137, 158)
> x1 <- c(-1, -1,  1,  1, -1, -1, 1, 1)
> x2 <- c(-1, -1, -1, -1,  1,  1, 1, 1)
> summary(lm(y ~ x1 + x2 + x1:x2))
> anova(lm(y ~ x1 + x2 + x1:x2))
> interaction.plot(x1, x2, y)
```

2^k Full Factorial Design

- **We will look at every possible combination of the two levels for *k* factors.**
- **Suppose our Catapult Experiment included 4 factors:**
  - **Angle: 180, Full**
  - **Peg Height: 3, 4**
  - **Stop Position: 3, 5**
  - **Hook Position: 3, 5**
- **Distance was measured for each of two runs <u>at each factor level combination</u>.**

Linear Model for this 2^4 Factorial Design

$$Y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i3} + \beta_4 x_{i4} + \beta_{12} x_{i1} x_{i2} + \beta_{13} x_{i1} x_{i3} + \beta_{14} x_{i1} x_{i4} + \beta_{23} x_{i2} x_{i3} + \beta_{24} x_{i2} x_{i4} + \beta_{34} x_{i3} x_{i4} + \beta_{123} x_{i1} x_{i2} x_{i3} + \beta_{124} x_{i1} x_{i2} x_{i4} + \beta_{134} x_{i1} x_{i3} x_{i4} + \beta_{234} x_{i2} x_{i3} x_{i4} + \beta_{1234} x_{i1} x_{i2} x_{i3} x_{i4} + \varepsilon_i$$

- **The formulas for the estimated effects can most easily be seen using a table of contrasts:**

1	-1	-1	-1	-1	1	1	1	1	1	1	-1	-1	-1	-1	1	(1)
1	1	-1	-1	-1	-1	-1	-1	1	1	1	1	1	1	-1	-1	a
1	-1	1	-1	-1	-1	1	1	-1	-1	1	1	1	-1	1	-1	b
1	1	1	-1	-1	1	-1	-1	-1	-1	1	-1	-1	1	1	1	ab
1	-1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	1	-1	c
1	1	-1	1	-1	-1	1	-1	-1	1	-1	-1	1	-1	1	1	ac
1	-1	1	1	-1	-1	-1	1	1	-1	-1	-1	1	1	-1	1	bc
1	1	1	1	-1	1	1	-1	1	-1	-1	1	-1	-1	-1	-1	abc
1	-1	-1	-1	1	1	1	-1	1	-1	-1	-1	1	1	1	-1	d
1	1	-1	-1	1	-1	-1	1	1	-1	-1	1	-1	-1	1	1	ad
1	-1	1	-1	1	-1	1	-1	-1	1	-1	1	-1	1	-1	1	bd
1	1	1	-1	1	1	-1	1	-1	1	-1	-1	1	-1	-1	-1	abd
1	-1	-1	1	1	1	-1	-1	-1	-1	1	1	1	-1	-1	1	cd
1	1	-1	1	1	-1	1	1	-1	-1	1	-1	-1	1	-1	-1	acd
1	-1	1	1	1	-1	-1	-1	1	1	1	-1	-1	-1	1	-1	bcd
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	abcd

```
                         trt.means
 [1,] -1 -1 -1 -1      358.5
 [2,]  1 -1 -1 -1      403.5
 [3,] -1  1 -1 -1      438.0
 [4,]  1  1 -1 -1      480.0
 [5,] -1 -1  1 -1      381.5
 [6,]  1 -1  1 -1      438.5
 [7,] -1  1  1 -1      475.5
 [8,]  1  1  1 -1      545.0
 [9,] -1 -1 -1  1      419.5
[10,]  1 -1 -1  1      484.0
[11,] -1  1 -1  1      491.0
[12,]  1  1 -1  1      537.5
[13,] -1 -1  1  1      456.5
[14,]  1 -1  1  1      531.0
[15,] -1  1  1  1      482.5
[16,]  1  1  1  1      585.5
```

• The estimated main effect of X_1 is thus:

• For two-level factorial designs, we get estimates for the main and interaction effects by dividing by half of the number of distinct factorial treatment combinations.
• Catapult example: $2^4 = 16$, so we divide by 8.

• Formulas for all other estimated main effects and interaction effects can be found using the appropriate column of the table of contrasts.

• In the linear model, the estimated coefficients are simply:

• **We again will always use software to get the estimates of these regression coefficients.**

R code for example:

```
> catapult4.data <-
read.table("http://www.stat.sc.edu/~hitchcock/CatapultFourFactor.
txt", header=TRUE)
> attach(catapult4.data)
> x1 <- angle; x2 <- PgHt; x3 <- SPos; x4 <- HPos;
y <- Dist
> as.vector(tapply(y, INDEX = list(x1, x2, x3, x4),
mean))
> summary(lm(y ~ x1 * x2 * x3 * x4))
```

Conclusions?

Advanced Concerns

• **If there is only one observation per treatment (no replication), formal tests about the effects are impossible.**

• **A graphical approach to determining which effects are significant involves a normal Q-Q plot of the estimated coefficients. Estimates deviating much from a straight line drawn through the effects near 0 correspond to "significant" effects.**

```
> qqnorm(coef(lm(y ~ x1 * x2 * x3 * x4))[-1],datax=T)
```

• **When <u>all</u> factors are continuous, the shape (linear? quadratic?) of the response surface can be ascertained by "center runs" (see p. 483-488).**