STAT 515 --- Chapter 3: Probability

Basic Definitions

Experiment: A process which leads to a single outcome (called a sample point) that cannot be predicted with certainty.

Sample Space (of an experiment): The collection of all the possible outcomes (or sample points).

Example 1. Roll 1 die:
Sample space =

Example 2. Toss 2 coins:
Sample space =

The probability of a sample point is a number between 0 and 1 that measures the likelihood that this outcome will occur when the experiment is performed.

Often we take this to mean the proportion of times the outcome would occur if we repeated the experiment many times.

Note:
(1) All sample point probabilities must be between 0 and 1.
(2) The probabilities of all the points in the sample space must sum to 1.

Example 1: Probability of rolling a 3, denoted:
P(3) =

Example 2: Assuming coin is fair, P(HH) =

An <u>event</u> is an outcome or collection of outcomes.

We typically determine the probability of an event by adding the probabilities of the distinct outcomes that make up the event.

Example 1: Event A = 'rolling an even number'

P(A) =

Example 2: Event B = 'get at least one head'

P(B) =

<u>Unions and Intersections</u>

<u>Compound events</u> are composed of two or more "simple events," for example:

- **The <u>union</u> of events A and B is the event that either A <u>or</u> B (or both) occurs.**
- **Denoted $A \cup B$**
- **The <u>intersection</u> of events A and B is the event that both A and B occur when the experiment is conducted.**
- **Denoted $A \cap B$**

<u>Venn Diagrams</u>: Represent graphically which sample points make up which events.

Pictures:

Example 1(b) (Roll 1 die): Define events:
A = {Roll an even number}
B = {Roll a number less than 4}

Venn Diagram:

$A \cup B =$

$A \cap B =$

$P(A \cup B) =$

$P(A \cap B) =$

Note that the idea of unions and intersections extend to situations with more than two events:

$A \cup B \cup C$ is the event that any of A, B, <u>or</u> C occurs.

$A \cap B \cap C$ is the event that A, B, <u>and</u> C occurs.

Complementary Events

The complement of an event A (denoted A^c) is the collection of outcomes that do not correspond to A.

Since all outcomes in the sample space are either in A or not in A (and thus are in A^c), then:

$$P(A) + P(A^c) = 1$$

Example 2(b): Suppose we toss 10 coins. Define event A = {obtain at least 1 "head"}.

What is P(A)?

Additive Rule of Probability

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Picture:

Example 3: (Deck of 52 cards:)

	Hearts	Non-Hearts
Face	3	9
Non-Face	10	30

We select 1 card randomly from the deck.

Define: A = {Card is face}
B = {Card is non-face}
C = {Card is heart}
D = {Card is non-heart}

<u>Mutually Exclusive Events</u>

Two or more events are <u>mutually exclusive</u> when the following is true: If one event occurs in an experiment, the other event cannot occur.

<u>Note</u>: Formally speaking, two events A and B are mutually exclusive if $P(A \cap B) = 0$. (That is, A and B have no outcomes in common.)

Example 3:
Are A and B mutually exclusive?

Are A and C mutually exclusive?

Conditional Probability

$P(A \mid B)$ = probability that event A occurs given that event B occurs when the experiment is conducted.

Recall die roll example:
A = {roll even number}, B = {roll number less than 4}

What is the probability of rolling an even number given that we roll a number less than 4?

Intuitively:

Formula: $$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \qquad \text{(assuming } P(B) \neq 0\text{)}$$

Example above:

Rearranging the Conditional Probability Formula, we get:

Multiplicative Rule of Probability:

$P(A \cap B) = P(B)\,P(A \mid B)$ **or** $P(A \cap B) = P(A)\,P(B \mid A)$

Example: Suppose the probability that my roof will leak when it rains is 0.3. Tomorrow, the probability of rain is 0.2. What is the probability that it rains <u>and</u> my roof leaks tomorrow?

Let A = {rain} and B = {roof leaks}

<u>Independent Events</u>: Events A and B are independent if the occurrence of B doesn't affect the probability that A occurs (and vice versa).

That is, A and B are independent <u>if and only if</u> $P(A \mid B) = P(A)$. **We could also say: A and B are independent if and only if** $P(B \mid A) = P(B)$.

Events that are not independent are called <u>dependent</u>.

<u>Intersection of Two Independent Events</u>:

If A and B are independent, then $P(A \cap B) = P(A)\,P(B)$. (Why?)
Conversely, if $P(A \cap B) = P(A)\,P(B)$, then A and B are independent.

<u>Die rolling example</u>:
A = {even number}, B = {less than 4}.
Are A and B independent?

But consider C = {less than 5}. Then A and C <u>are</u> independent. Why?

More Conditional Probability:

Example: Suppose 78% of suspect drivers get a breath test, 36% a blood test, and 22% get both. What is the probability that a driver who does get a breath test also gets a blood test?

Note: "Independent" and "Mutually exclusive" are very different concepts.

• If two events A and B are mutually exclusive (and $P(A) > 0$ and $P(B) > 0$), then they cannot be independent. Why?

<u>Bayes' Rule</u>:

$$P(B \mid A) = \frac{P(B \cap A)}{P(A)} \quad =$$

→ Bayes' Formula:

• Allows us to express P(B | A) in terms of P(A | B), which may be easier to work with.

Example: A virus infects 1 out of every 200 people. The test to detect the virus is positive 80% of the time if a person has the virus, and positive 5% of the time if the person <u>does not</u> have the virus.

• If a person tests positive, what is the probability that person is infected?

• **If a person tests negative, what is the probability that person is <u>not</u> infected?**

Bayes' Rule can be extended to a situation where we have $k > 2$ mutually exclusive events $B_1, B_2, \ldots, B_k$ such that $P(B_1) + \ldots + P(B_k) = 1$. Given that event A occurs, the probability of an event B_i occurring (where $i = 1, 2, \ldots, k$) is:

Random Sampling:

• In statistical inference, we make conclusions about the population based on the sample.

• To assign precise probabilities to our conclusions, the sample must be representative of the population.

• Best way to assure this: Use a <u>simple random sample</u>.

• A <u>simple random sample</u> (SRS) of *n* objects is chosen so that every possible set of *n* objects has an equal chance of being chosen.

• If the population can be listed, then we can use a random number table or computer to randomly select the items that will make up the sample.

• The methods we will learn in this class ASSUME that the data come from a random sample of the population of interest.

<u>Samples that are NOT simple random samples</u>:
• If we select every 20th name in the USC student directory, this does not constitute a SRS of USC students.
• If we randomly pick several counties in South Carolina and interview every adult in the selected counties, this does not constitute a SRS of South Carolina adults.