Multi-factor Factorial Experiments

● In the one-way ANOVA, we had a single factor having several different levels.

● Many experiments have multiple factors that may affect the response.

Example: Studying weight gain in puppies

Response (Y) = weight gain in pounds

Factors:

● Here, 3 factors, each with several levels.

● Levels could be quantitative or qualitative.

● A factorial experiment measures a response for each combination of levels of several factors.

● Example above is a:

● We will study the effect on the response of the factors, taken individually and taken together.

Two Types of Effects

● The main effects of a factor measure the change in mean response across the levels of that factor (taken individually).

● Interaction effects measure how the effect of one factor varies for different levels of another factor.

Example: We may study the main effects of food amount on weight gain.

● But perhaps the effect of food amount is different for each type of diet: Interaction between amount and diet!

Picture:

Two-Factor Factorial Experiments

● **Model is more complicated than one-way ANOVA model.**

● **Assume we have two factors, A and C, with *a* and *c* levels, respectively:**

● **Assume we have *n* observations at each combination of factor levels.**

● **Total of observations.**

Model:

● Y_{ijk} **=** ***k*-th observed response at level *i* of factor A and level *j* of factor C.**
● μ **= an overall mean response**
● α_i**'s (main effects of factor A) = difference between mean response for *i*-th level of A and the overall mean response**
● γ_j**'s (main effects of factor C) = difference between mean response for *j*-th level of C and the overall mean response**
● $(\alpha\gamma)_{ij}$**'s (interaction effects between factors A and C)**
● ε_{ijk} **= random error component → accounts for the variation among responses <u>at the same combination</u> of factor levels**

● **Again, we assume the random error is approximately normal, with mean 0 and variance σ^2.**

● **We also restrict** $\sum_i \alpha_i = \sum_j \gamma_j = \sum_i (\alpha\gamma)_{ij} = \sum_j (\alpha\gamma)_{ij} = 0.$

Example: (Meaning of main effects)

● **Suppose $\alpha_1 = 3.5$ and $\alpha_2 = 2$. What does this mean?**

Case I: (No interaction between A and C)
→ $(\alpha\gamma)_{ij} = 0$ for all *i, j*

● **Mean response at level 1 of factor A is:**

● **Mean response at level 2 of factor A is:**

● **For any fixed level of C, mean response at level 1 of A**

Picture:

Case II: (Interaction between A and C)

- **Mean response at level 1 of factor A is:**

- **Mean response at level 2 of factor A is:**

- **Here, the difference in mean responses for levels 1 and 2 of factor A is:**

- **This difference depends on the level of C!**

Picture:

- **We see that the main effects are not directly interpretable in the presence of interaction.**

● In a two-factor study, first we will test for interaction:

● If there is no significant interaction, we will test for main effects of each factor:

Notation for Sample Means:

$\bar{Y}_{ij\bullet}$ **= sample mean of observations for level *i* of A and level *j* of C [This is the (*i, j*) cell sample mean]**

$\bar{Y}_{i\bullet\bullet}$ **= sample mean of observations for level *i* of A**

$\bar{Y}_{\bullet j\bullet}$ **= sample mean of observations for level *j* of C**

$\bar{Y}_{\bullet\bullet\bullet}$ **= sample mean of all observations in the study [This is the <u>overall</u> sample mean]**

ANOVA Table for Two-Factor Experiment

● Partitioning the Variation in Y:

TSS =

SS(Cells) =

SSW =

Picture:

MS(Cells) = **MSW =**

● If MS(Cells) > MSW, the mean response is different across the cells → the ANOVA model is not useless.

<u>Overall F-test</u>: If F* = MS(Cells) / MSW is greater than $F_{\alpha}[ac - 1, ac(n - 1)]$, then we conclude there is a difference among the population cell means.

<u>Example (Table 9.5 data)</u>:

● Software will calculate:

F* =

Using $\alpha = 0.05$:

Conclusion:

● **If we reject H_0: "all cell means are equal" with the overall F-test, then we test for (1) interaction and possibly (2) main effects.**

● **Further Partitioning of SS(Cells):**

SSA = $cn\sum_i (\bar{Y}_{i\bullet\bullet} - \bar{Y}_{\bullet\bullet\bullet})^2$ **d.f. = *a* – 1**

→

SSC = $an\sum_j (\bar{Y}_{\bullet j\bullet} - \bar{Y}_{\bullet\bullet\bullet})^2$ **d.f. = *c* – 1**

→

SSAC = SS(Cells) – SSA – SSC **d.f. = (*a* – 1)(*c* – 1)**

→

Mean Squares:

MSA = **MSC =** **MSAC =**

ANOVA table

Source	d.f.	SS	MS	F*

● We will usually calculate the ANOVA table quantities using software.

Useful F-tests in Two-Factor ANOVA

Testing for Significant Interaction: We reject

H_0: $(\alpha\gamma)_{ij} = 0$ for all *i, j*

if:

Example:

Note: If (and only if) the interaction is NOT significant, we test for significant main effects of factor A and of factor C:

● For factor A: We reject H_0: $\alpha_i = 0$ for all *i*
if:

● For factor C: We reject H_0: $\gamma_j = 0$ for all *j*
if:

Interpreting a Significant Interaction

- **Generally done by examining Interaction Plots.**

Example (Gas mileage data):

Conclusions: