Sec. 3.5 --- Variations of the Sign Test

• **Tests based on the sign test can be used to answer a variety of questions.**

McNemar's Test

• **Consider two paired binary variables (nominal).**

• **Both X and Y can only take the values 0 and 1, say.**

• **This type of data often arises from "before vs. after" experiments.**

• **$X = 1$ might represent having some condition** before **a treatment is applied and $Y = 1$ having it** after **the treatment is applied.**

• **Question: Is the probability of having the condition the same before and after the treatment is applied?**

• **Or does the treatment change the probability of having it?**

Null and Alternative hypotheses:

$$H_0: P(X_i=1) = P(Y_i=1) \text{ for all } i \quad \text{vs. } H_1: P(X_i=1) \neq P(Y_i=1) \text{ for all } i$$

$\leftrightarrow$

$$H_0: P(X_i=1, Y_i=1) + P(X_i=1, Y_i=0) = P(X_i=1, Y_i=1) + P(X_i=0, Y_i=1)$$

$$\Leftrightarrow H_0: P(X_i=1, Y_i=0) = P(X_i=0, Y_i=1)$$

• The data from such a study can be summarized with a 2×2 table:

		Y_i	
		0	1
X_i	0	a	b
	1	c	d

a, b, c, d are the counts of observations falling in each cell.

• Consider the ($X_i = 0$, $Y_i = 1$) entries to be the "+" observations.

• Let the ($X_i = 1$, $Y_i = 0$) entries be the "–" observations.

• Then we can use the <u>sign</u> test to test H_0.

• Note the *a* and *d* entries are treated as <u>ties</u>.

• The test statistic is simply $T_2 = b$ (the number of +'s)

• The null distribution of T_2 is binomial$(n = b+c,\ p = 0.5)$

Using Table A3 with $n = b + c$ and $p = 0.5$, reject H_0 if

$$T_2 \le t \quad \text{or} \quad T_2 \ge n - t$$

where t is the value corresponding to a probability of $\approx \alpha/2$.

The P-value is

$$2\left[\min\left\{P(T_2 \le t_2^{obs}),\ P(T_2 \ge t_2^{obs})\right\}\right]$$

Example 1: Suppose 200 subjects were asked last month and again this month whether they approved of the president's job performance. 90 said "yes" both times; 90 said "no" both times; 12 said "yes" the first month and "no" the second, and 8 said "no" the first month and "yes" the second. At $\alpha = 0.05$, has the president's approval rating significantly changed?

Contingency Table: Let no = 0, yes = 1

X = first month
Y = next month

X \ Y	0	1
0	90	8
1	12	90

Hypotheses: $H_0: P(X_i = 1) = P(Y_i = 1)$ for all i

$H_1: P(X_i = 1) \neq P(Y_i = 1)$ for all i

Test Statistic: $T_2 =$ 8

Reject Ho if $T_2 \le 5$ or $T_2 \ge 20-5$

Look at binomial table with $n = 20$, $p = 0.5$

$P(Y \le 5) = .0207$

P-value:

$2[\min\{P(T_2 \le 8), P(T_2 \ge 8)\}] = 2[\min\{.2517, 1-.1316\}] = \boxed{.5034}$

Conclusion:

At $\alpha = .05$, fail to reject Ho. We cannot conclude the approval rating significantly changed.

• **For large samples ($n > 20$), we can use the test statistic**

$$T_1 = \frac{(b-c)^2}{b+c}$$

which has a χ^2_1 **null distribution.** ← approximately, for large samples.

Why is this? $T_2 = b$ is binomial$(b+c, 0.5)$ under H_0. Then, if $b+c$ is large, we can approximate this with a normal distribution with mean $(0.5)(b+c)$ and standard deviation $\sqrt{(b+c)(0.5)(0.5)}$.

So
$$Z = \frac{T_2 - \frac{b+c}{2}}{\sqrt{(\frac{1}{2})(\frac{1}{2})(b+c)}} = \frac{b - \frac{1}{2}b - \frac{1}{2}c}{\frac{1}{2}\sqrt{b+c}} = \frac{\frac{1}{2}(b - c)}{\frac{1}{2}\sqrt{b+c}}$$

and $Z = \frac{b-c}{\sqrt{b+c}}$ has an approx. $N(0,1)$ distribution,

$\Rightarrow Z^2 = \frac{(b-c)^2}{b+c}$ has approx. χ^2_1 distn.

Cox-Stuart Test for Trend

• **An ordered sequence of numbers exhibits <u>trend</u> if the later numbers in the sequence tend to be greater than the earlier numbers (** increasing **trend) or if the later numbers in the sequence tend to be less than the earlier numbers (** decreasing **trend).**

• **In the arranged data, we essentially pair points to the left of the middle ordered value with points to the right of the middle ordered value, and perform a sign test.**

• **We assume the data $X_1, X_2, \ldots, X_{n'}$ are at least ordinal in scale.**

• **Pair the data as $(X_1, X_{1+c}), (X_2, X_{2+c}), \ldots, (X_{n'-c}, X_{n'})$ where $c = n'/2$ if n' is even; $c = (n'+1)/2$ if n' is odd.**

• **Note that if n' is odd, the middle value is ignored.**

• **If the first element in a pair is less than the second, we write a "+" for that pair.**
• **If the first element in a pair is greater than the second, we write a "–" for that pair.**
• **If the first element in a pair equals the second, we ignore that pair.**

Null hypothesis: H_0: no trend

3 possible alternatives:

Two-tailed	Lower-tailed	Upper-tailed
H_1: either increasing or decreasing trend	H_1: decreasing trend	H_1: increasing trend

Test Statistic: T = total number of +'s.

Null distribution of T is binomial **with p =** 0.5 **and n = the number of <u>untied</u> pairs.**

• **The decision rule and p-value are obtained in the same way as the sign test.**

Example: **NASA data give the average global temperatures for the last 13 decades, from the 1880s to the 2000s:** `-0.493, -0.457, -0.466, -0.497, -0.315, -0.077, 0.063, -0.036, -0.025, -0.002, 0.317, 0.563, 0.923.` **(Temperatures given in degrees F, centered by subtracting from 1951-1980 mean). Does Cox-Stuart test find evidence (at $\alpha = 0.05$) of an increasing trend?**

Hypotheses: H_0: no increasing trend H_1: increasing trend

$n' =$ 13 **and** $c = \frac{13+1}{2} = 7$

Pairs: $(X_1, X_8), (X_2, X_9), (X_3, X_{10}) (X_4, X_{11}) (X_5, X_{12}) (X_6, X_{13})$

$(-.493, -.036)\ (-.457, -.025), (-.466, -.002) (-.497, .317) (-.315, .563)\ (-.077, .923)$

$+ \quad + \quad + \quad + \quad + \quad +$

$T =$ 6

Look at binomial table with $p = 0.5$ **and** $n =$ 6

P-value = $P(T \geq 6) = 1 - P(T \leq 5) = 1 - .9844 = \boxed{.0156}$

Conclusion: We reject Ho. We can conclude an increasing trend in global temperature.

• **In order to test for a specified type of trend other than increasing or decreasing (such as periodic, alternating, etc.), the data must first be reordered to reflect the expected ordering according to the specified trend. Then the Cox-Stuart test can be implemented on the reordered data.**