Chapter 4: Continuous Random Variables and Probability Distributions

- Continuous r.v.'s take an uncountably infinite number of possible values.

Defn: The support of a r.v. is the set of values for which the r.v. has positive probability.

- Continuous r.v.'s have their support on an interval (or possibly on several intervals).

Examples:
- Heights of people
- Weights of apples
- Diameters of bolts
- Life lengths of lightbulbs

Note: Although when we measure these r.v.'s, we may round to a certain number of decimal places, inherently they could take any value in some interval.

- We cannot assign positive probabilities to every particular point in an interval, so with continuous r.v.'s, probability distributions are described and used differently.

4.2 Continuous Probability Distributions

- We first consider <u>any</u> r.v. (discrete or continuous).

<u>Defn (cdf)</u>: The <u>(cumulative) distribution function</u> (or cdf) of a r.v. Y is denoted $F(y)$ and is defined as

- Note that $F(y)$ is a function defined everywhere on the real line, regardless of which values of Y are possible.

<u>Example</u> (cdf of a discrete r.v.):
Let Y be a discrete r.v. with probability function

y	1	2	3
$p(y)$	0.3	0.5	0.2

Find the cdf of Y and sketch the graph of the cdf.

So:

Interval:	
$F(y)$:	

Sketch of graph of $F(y)$:

Note: The cdf for any discrete r.v. is a __________ function.

- There are a finite or countable number of points where $P(Y \leq y)$ increases.

Properties of a cdf

(1)

(2)

(3)

Example of (3):

(4)

The cdf of a continuous r.v.

- The cdf of a continuous r.v. is a continuous increasing function of y that is smooth over some interval of the real line.

__Defn.__ A r.v. Y is __continuous__ if its cdf $F(y)$ is continuous for $-\infty < y < \infty$.

- As y increases, $F(y) = P(Y \leq y)$ increases "gradually," not in steps.
- Note that with a continuous r.v. Y, for any real number y,

- Otherwise, if any value had positive probability, the cdf would "step up" at that value and Y would not be continuous.
- The probability function of a discrete r.v. is sometimes called a __probability mass function__ (pmf) since it places a "mass" of probability on certain values.
- For continuous r.v.'s the pmf is replaced by a __probability density function__ (pdf).

Defn. Suppose Y is a continuous r.v. with cdf $F(y)$. Then the probability density function of Y is

wherever this derivative exists.

Corollary: We see that

Properties of a pdf

(1) $f(y) \geq 0$ for all y (pdf always nonnegative)

Proof:

(2) $\int_{-\infty}^{\infty} f(t)\,dt = 1$ (pdf integrates to 1)

Proof:

Example 1: Let Y be a continuous r.v. with cdf:

$$F(y) = \begin{cases} 0 & \text{for } y < 0 \\ y^3 & \text{for } 0 \le y \le 1 \\ 1 & \text{for } y > 1 \end{cases}$$

Find the pdf $f(y)$ and graph both $F(y)$ and $f(y)$. Find $P(Y \le 0.5)$.

Graph of $F(y)$:

Graph of $f(y)$:

$P(Y \le 0.5) =$

Finding Arbitrary Probabilities

- With continuous r.v.'s, we usually want the probability that Y falls in a specific interval $[a,b]$; i.e., we want $P(a \leq Y \leq b)$ for numbers $a < b$.
- <u>Theorem</u>: If Y is a continuous r.v. with pdf $f(y)$ and $a < b$, then

<u>Note</u>: This represents the "area under the density curve" between $y=a$ and $y=b$:

Picture:

<u>Proof</u>:

Example 1 again: Find $P(0.4 \le Y \le 0.8)$.

Note this is simply

Picture:

- We could consider intervals of infinite length, where $a = -\infty$ and/or $b = \infty$.

Note: For a continuous r.v. Y and $a<b$,

Defn: (Quantile) Let $0<p<1$. The p-th quantile of a r.v. Y (denoted ϕ_p) is the smallest value such that

If Y is continuous, ϕ_p is the smallest value such that

Note: The median of a r.v. Y is $\phi_{.50}$, the 0.50 quantile (or 50th percentile) of Y.

Example 1 again: Find the median of Y.

Graph of $f(y)$ for Example 1:

Example 2: Let Y be a continuous r.v. with pdf

$$f(y) = \begin{cases} c y^{-1/2} & \text{for } 0 < y \leq 4 \\ 0 & \text{elsewhere} \end{cases}$$

for some constant c.

Find c, find $F(y)$, graph $f(y)$ and $F(y)$, and find $P(Y < 1)$.

We know

-Since Y has support only on

Clearly

Graph of $f(y)$:

Graph of $F(y)$:

$P(Y < 1) =$

Also note