5.3 Marginal and Conditional Distributions

- A marginal distribution associated with a random vector gives the distribution for one of the set of r.v.'s constituting $\underset{\sim}{Y}$.
- For a discrete random vector (Y_1, Y_2), the marginal distribution for one r.v. is found by summing joint probabilities across the possible values of the other r.v.

Defn. If Y_1, Y_2 are jointly discrete r.v.'s with joint pmf $p(y_1, y_2)$, then the marginal probability function of Y_1 is:

- Similarly, the marginal probability function of Y_2 is:

Example 3 again: Find the marginal distribution of Y_1, the number of contracts awarded to USC.

- So the marginal distribution of Y_1 is:

- We could sum across y_1 values to get the marginal distribution of Y_2:

- Note that the marginal distributions are represented as the margins of the two-way table.
- For jointly continuous r.v.'s, we find a marginal distribution by integrating over the possible values of the other r.v.

Defn. If Y_1, Y_2 are jointly continuous r.v.'s with joint pdf $f(y_1, y_2)$ then the marginal pdf of Y_1 is:

and the marginal pdf of Y_2 is:

Example 6: Suppose we model two proportions Y_1 and Y_2 with the joint pdf

$$f(y_1, y_2) = \begin{cases} 0.4(y_1 + 4y_2) & \text{for } 0<y_1<1,\ 0<y_2<1 \\ 0 & \text{elsewhere} \end{cases}.$$

- Find the marginal pdf of Y_1 and find $P(Y_1 > 0.3)$.

$P(Y_1 > 0.3) =$

-Example 7: Suppose for two proportions Y_1 and Y_2, we know $Y_1 < Y_2$. We assume the joint pdf:

$$f(y_1, y_2) = \begin{cases} 6y_1 & \text{for } 0 < y_1 < y_2 < 1 \\ 0 & \text{elsewhere} \end{cases}$$

- Find and identify the marginal distributions for Y_1 and for Y_2.

-So the marginal pdf of Y_1 is a __________ pdf.

-So the marginal pdf of Y_2 is a __________ pdf.

Conditional Distributions

- In the discrete case, we may define conditional probability distributions based on the definition of conditional probability.

__Defn:__ If Y_1 and Y_2 are jointly discrete r.v.'s with joint pmf $p(y_1, y_2)$ and marginal pmf's $p_1(y_1)$ and $p_2(y_2)$, then the __conditional probability function__ of Y_1 given Y_2 is:

provided that

- Similarly, the __conditional probability function__ of Y_2 given Y_1 is:

__Example 3 again:__ Find the conditional distribution of Y_1 given $Y_2 = 0$ and find $P(Y_1 \geq 1 \mid Y_2 = 0)$.

- In the joint pmf table, we focus on the row where

- Now consider jointly continuous r.v.'s Y_1 and Y_2.
- Recall the joint pdf $f(y_1, y_2)$ is a 3-D surface on the (y_1, y_2) plane:

- If we consider the conditional pdf $f(\cdot \mid y_2^*)$ at a particular value y_2^* of Y_2, we are taking a "slice" (cross-section) of this surface.

- This slice intersects the surface in the curve

and the area under that curve is

and thus is a proper pdf which integrates to 1.

<u>Defn:</u> In general, if Y_1 and Y_2 are jointly continuous with joint pdf $f(y_1, y_2)$ and marginal pdf's $f_1(y_1)$ and $f_2(y_2)$, then the <u>conditional pdf</u> of Y_1 given Y_2 is:

- Similarly, the conditional pdf of Y_2 given Y_1 is:

<u>Example 7 again:</u> Find $f(y_2 | y_1)$, find $P(Y_2 < 0.8 | Y_1 = 0.4)$, and find $P(Y_2 < 0.2 | Y_1 = 0.4)$.

What if the conditional probability we wanted was something like

$$P(Y_2 < 0.5 \mid Y_1 > 0.5) \quad ?$$

- Use the basic idea of conditional probability to write it as:

- Can find numerator probability by integrating the joint pdf $f(y_1, y_2)$:

- Can find denominator probability by integrating the marginal pdf of Y_1, $f_1(y_1)$:

<u>Exercise</u>: Show the conditional probability above is

- Here, we treat y_1 as a constant. We are modeling the behavior of Y_2, given $Y_1 = y_1$.
- Note that $Y_2 | Y_1 = y_1$ follows a __________ distribution.
- So $P(Y_2 < 0.8 | Y_1 = 0.4) =$

And note $P(Y_2 < 0.2 | Y_1 = 0.4) =$ ______ since the support of $Y_2 | Y_1 = 0.4$ is the interval ______.

Example 6 again: Find $f(y_2 | y_1)$ and find $P(Y_2 < 0.5 | Y_1 = 0.5)$.

5.4 Independent Random Variables

- Recall in Example 7 we showed
  $P(Y_2 < 0.8 \mid Y_1 = 0.4) \approx$
- Marginally, $Y_2 \sim \text{beta}(3,1)$ and so it can be easily shown that $P(Y_2 < 0.8) \approx$
- Thus knowledge of the exact value of Y_1 altered the probability that $Y_2 < 0.8$.
- This indicates that Y_1 and Y_2 are ________ r.v.'s.
- In general, for any real numbers $a < b$ and $c < d$, our earlier definition of independent events implies

if Y_1, Y_2 are independent.

- This can be formalized:

Defn. Let Y_1 and Y_2 be r.v.'s with cdf's $F_1(y_1)$ and $F_2(y_2)$ and joint cdf $F(y_1, y_2)$. Then Y_1 and Y_2 are independent if and only if

- Otherwise, Y_1 and Y_2 are __________.

<u>Theorem</u> (Independence of discrete r.v.'s)
If Y_1 and Y_2 are discrete r.v.'s with pmf's $p_1(y_1)$ and $p_2(y_2)$ and joint pmf $p(y_1, y_2)$, then Y_1 and Y_2 are <u>independent</u> if and only if

for <u>all</u> pairs (y_1, y_2).

<u>Theorem</u> (Independence of continuous r.v.'s)
If Y_1 and Y_2 are continuous r.v.'s with pdf's $f_1(y_1)$ and $f_2(y_2)$ and joint pdf $f(y_1, y_2)$, then Y_1 and Y_2 are independent if and only if

for all pairs (y_1, y_2).

<u>Corollary</u>: Y_1 and Y_2 are independent if and only if any of the following hold:

(1)
(2)
(3)
(4)
(5)

- Conditions (1)-(5) are equivalent, and checking any of these suffices to prove or disprove independence.

<u>Example 6 again</u>: Are Y_1 and Y_2 independent?

<u>Example 5 again</u>: Are Y_1 and Y_2 independent?

Recall

$$f(y_1, y_2) = \begin{cases} e^{-(y_1+y_2)} & \text{for } y_1>0,\ y_2>0 \\ 0 & \text{elsewhere} \end{cases}$$

So

Helpful Theorem: If the joint pdf for Y_1 and Y_2 can be factored into two nonnegative functions, i.e.,

where $g(y_1)$ is a function of only y_1 and $h(y_2)$ is a function of only y_2, then Y_1 and Y_2 are independent.

Note: $g(y_1)$ and $h(y_2)$ need not be valid pdf's, although they will be proportional to valid pdf's.

Example 8: Let Y_1 and Y_2 have joint pdf

$$f(y_1, y_2) = \begin{cases} 2y_1, & 0 \le y_1 \le 1,\ 0 \le y_2 \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

Then $f(y_1, y_2) = g(y_1)\,h(y_2)$ where

Note

- Note that if the support of the joint pdf is such that y_1 is constrained by y_2 (or y_2 is constrained by y_1), then Y_1 and Y_2 are dependent.

Example 7 again:

$$f(y_1, y_2) = \begin{cases} 6y_1 & \text{for } 0 < y_2 < y_1 < 1 \\ 0 & \text{elsewhere} \end{cases}$$

- From the support, we see that y_2 is constrained by y_1 (and y_1 by y_2), so Y_1 and Y_2 are dependent.

Theorem (Independence of n r.v.'s):

- The r.v.'s $Y_1, Y_2, \ldots, Y_n$ are independent if and only if

where $f_i(y_i)$ is the pdf of the i-th r.v., $i = 1, \ldots, n$.

- If we take a random sample of n measurements from a population, the n r.v.'s are said to be independent and identically distributed (iid).

<u>Example</u>: Lifelengths (in hours) of a population of batteries follow an expon(30) distribution. We take a random sample of 5 batteries and will observe their lifelengths. Find the joint pdf of the 5 measurements.