5.5 The Expected Value of a Function of a Random Vector

Defn: Let $(Y_1, Y_2, \dots, Y_k)$ be jointly discrete r.v.'s having joint pmf $p(y_1, y_2, \dots, y_k)$. Then for some function $g(Y_1, Y_2, \dots, Y_k)$:

- Similarly, for jointly continuous r.v.'s $(Y_1, \dots, Y_k)$ with joint pdf $f(y_1, \dots, y_k)$:

- These expected values exist if the multiple sum (integral) is absolutely convergent.

Example 9: Consider a product that contains impurities, some toxic and some nontoxic. Let Y_1 = the proportion of a sample of the product that is impure. Let Y_2 represent the proportion of the impurities that are toxic. The joint pdf of Y_1 and Y_2 is:

$$f(y_1, y_2) = \begin{cases} 2(1-y_1), & 0 \le y_1 \le 1, \quad 0 \le y_2 \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find the expected proportion of the sampled product containing toxic impurities.

We want

- Given the joint pdf, we can find expected values of functions of Y_1 alone or Y_2 alone as well.

Example: $E(Y_1) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y_1 f(y_1, y_2)\, dy_1\, dy_2$

[this is the same as $\int_{-\infty}^{\infty} y_1 f_1(y_1)\, dy_1$, where $f_1(y_1)$ is the marginal pdf of Y_1].

Similarly: $E(Y_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y_2 f(y_1, y_2)\, dy_1\, dy_2$,

$V(Y_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [y_2 - E(Y_2)]^2 f(y_1, y_2)\, dy_1\, dy_2$, etc.

5.6 Special Expected Values

Theorem: If c is a constant, then $E(c) = c$.

Theorem: If $g(Y_1, Y_2)$ is a function of r.v.'s Y_1 and Y_2, and c is a constant, then:

Theorem: If $g_1(Y_1, Y_2), \ldots, g_k(Y_1, Y_2)$ are several functions of the r.v.'s Y_1 and Y_2, then:

- The proofs are similar to those in the univariate case.

Example 7 again: Recall Y_1 and Y_2 have joint pdf

and marginally, $Y_1 \sim \text{beta}(2,2)$ and $Y_2 \sim \text{beta}(3,1)$.
Find $E(Y_2 - Y_1)$.
One approach is to integrate using the joint pdf:

Or note:

Theorem: If Y_1, Y_2 are independent r.v.'s and $g(Y_1)$ is a function of Y_1 alone and $h(Y_2)$ is a function of Y_2 alone, then

provided the expected values exist.

Proof (continuous case):

- The proof is similar in the discrete case.

Corollary: If Y_1 and Y_2 are independent, then $E(Y_1Y_2) = E(Y_1)E(Y_2)$.

Proof:

Example 9 again: It is easy to see (by factoring the joint pdf) that Y_1 and Y_2 are independent and have marginal pdf's

- We can easily then find $E(Y_1Y_2)$ by showing $E(Y_1)E(Y_2) = \left(\frac{1}{3}\right)\left(\frac{1}{2}\right) = \frac{1}{6}$. (Verify as an exercise)

5.7 Covariance of Two Random Variables

- The <u>covariance</u> between Y_1 and Y_2 measures how the two variables tend to <u>vary</u> <u>together</u> (in a <u>linear</u> manner).

<u>Defn</u>: If Y_1 and Y_2 are r.v.'s with respective means μ_1 and μ_2, then:

$$\text{cov}(Y_1, Y_2) =$$

<u>Note</u>: $\text{cov}(Y_1, Y_2) =$

since

$\text{cov}(Y_1, Y_2) > 0$ implies Y_1 and Y_2 are <u>positively</u> linearly related (as Y_1 increases, Y_2 tends to increase).

$\text{cov}(Y_1, Y_2) < 0$ implies Y_1 and Y_2 are <u>negatively</u> linearly related (as Y_1 increases, Y_2 tends to decrease, and vice versa).

$\text{cov}(Y_1, Y_2) = 0$ implies no <u>linear</u> association between Y_1 and Y_2.

- <u>Note</u>: It is clear from the definition that

$$\text{cov}(Y_1, Y_1) =$$

- It is hard to judge the <u>strength</u> of the linear association using the covariance since the covariance depends on the scale of measurement.
- We often use a <u>standardized</u> version called the <u>correlation coefficient</u> (denoted by ρ):

- It is always true that $-1 \le \rho \le 1$, so values of ρ near 1 (or near -1) indicate a <u>strong</u> positive (or negative) linear association.

<u>Lemma</u> (Cauchy-Schwarz Inequality):
Let U and V be r.v.'s with $E(U^2) < \infty$ and $E(V^2) < \infty$. Then

- This is a particular case of a standard result from calculus.

<u>Proof that $-1 \le \rho \le 1$:</u> Let

<u>Example 7 again:</u> $f(y_1, y_2) = \begin{cases} 6y_1, & 0<y_1<y_2<1 \\ 0 & \text{elsewhere} \end{cases}$

Find $\text{cov}(Y_1, Y_2)$ and find ρ.

- We have seen that marginally, $Y_1 \sim \text{beta}(2,2)$ and $Y_2 \sim \text{beta}(3,1)$ and so

- So there is a __________ __________ <u>linear</u> association between Y_1 and Y_2.

Theorem (Independence and Covariance):
If Y_1 and Y_2 are independent, then
(and thus).

Proof:

- The converse is not true: Two r.v.'s could have covariance zero yet still be dependent.

Example: Let $Y_1 \sim Unif(-1,1)$ and let $Y_2 = Y_1^2$. Then