6.4 The Method of Transformations

- Can be used to find the pdf of $U = h(Y)$ <u>if</u> $h(\cdot)$ is an increasing function <u>or</u> a decreasing function (one-to-one function).
- If so, the inverse function $h^{-1}(\cdot)$ exists.

Suppose (WLOG) that $h(\cdot)$ is an increasing function and $U = h(Y)$, where Y has pdf $f_Y(y)$.

Then $\{ y : h(y) \leq u^* \} \equiv$

<u>Picture:</u>

Method of Transformations Steps:

(1) Verify $h(\cdot)$ is an increasing function or a decreasing function.

(2) Write the inverse function $y = h^{-1}(u)$.

(3) Find the derivative $\frac{dy}{du}$ of this inverse function.

(4) Then the density of $U = h(Y)$ is:

$$f_U(u) = f_Y(y)\left|\frac{dy}{du}\right|,$$ plugging in $h^{-1}(u)$ for y.

Example 1: Let Y be a standard exponential r.v., i.e.,

$$f_Y(y) = \begin{cases} e^{-y} & \text{if } y > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Find the pdf of $U = \sqrt{Y}$.

- First note $h(y) = \sqrt{y}$ is
- The inverse function is:

Probability Integral Transformation

Example 2: Suppose $F(y)$ is the cdf of a continuous r.v. Y. Find the density of $U = F(Y)$.

- We know the cdf of a continuous r.v. is

- Clearly $\frac{du}{dy}$ is simply

- But what is $\frac{dy}{du}$? Note: $\frac{dy}{du} =$

Example 3: Let $Z \sim N(0,1)$. Find the pdf of $U = Z^2$.

Problem! $h(z) = z^2$ is <u>not</u> monotone on $(-\infty, \infty)$!

Picture:

Solution: Break support of Z up into two pieces, then use transformation technique on each one.

On $A_1 = (-\infty, 0)$, $h(\cdot)$ is

On $A_2 = [0, \infty)$, $h(\cdot)$ is

On A_1,

On A_2,

- To get the pdf of U, we can <u>add</u> the terms coming from each piece, A_1 and A_2.

Example 4: Let $Y \sim \text{Unif}\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Find the pdf of $U = \tan(Y)$.

- Note $h(y) = \tan(y)$ is
- The inverse function is:

- This is called a standard Cauchy distribution.
- Fact: If U is a Cauchy r.v., then $\int_{-\infty}^{\infty} |u| f_U(u)\, du$ is not finite. Hence $E(U)$ does not exist.

Plot of Cauchy pdf:

6.6 Bivariate Transformation Technique

- Now we consider a <u>bivariate</u> <u>transformation</u> of jointly continuous r.v.'s Y_1 and Y_2. Let

$$U_1 = h_1(Y_1, Y_2)$$
$$U_2 = h_2(Y_1, Y_2)$$

where the transformation of (Y_1, Y_2) into (U_1, U_2) is one-to-one, i.e., there are unique inverse functions $y_1 = h_1^{-1}(u_1, u_2)$ and $y_2 = h_2^{-1}(u_1, u_2)$.

If $f_{Y_1,Y_2}(y_1, y_2)$ is the joint pdf of (Y_1, Y_2), then the joint pdf of (U_1, U_2) is:

plugging in $h_1^{-1}(u_1, u_2)$ for y_1 and $h_2^{-1}(u_1, u_2)$ for y_2.

Here $|J|$ is the absolute value of the Jacobian, which is:

Recall the determinant of a 2×2 matrix can be found by: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} =$

Note: We must pay attention to what the support of the new r.v.'s (U_1, U_2) will be!

- If we want the marginal distribution of U_1 or of U_2, we can simply integrate the joint distribution:

$$f_{U_1}(u_1) =$$

or

$$f_{U_2}(u_2) =$$

Example 1: Suppose Y_1 and Y_2 are r.v.'s with joint density

$$f_{Y_1,Y_2}(y_1,y_2) = \begin{cases} e^{-(y_1+y_2)} & \text{if } y_1>0,\ y_2>0 \\ 0 & \text{elsewhere} \end{cases}$$

Find the joint density of $U_1 = Y_1 + Y_2$ and $U_2 = \dfrac{Y_1}{Y_1+Y_2}$.

Solve for inverse functions:

Note: Can show the marginal density of $U_2 = \frac{Y_1}{Y_1 + Y_2}$ is Unif $(0,1)$. Integrate out U_1:

Exercise: Show $\int_0^\infty u_1 e^{-u_1} du_1 = \Gamma(2) = 1$

using properties of gamma pdf or integration by parts.

$\Rightarrow f_{U_2}(u_2) =$

Conclusion: If Y_1, Y_2 are independent standard exponential r.v.'s, then $\frac{Y_1}{Y_1 + Y_2}$ is standard uniform.

<u>Example 2</u>: You independently choose two values at random between 0 and 1. What is the distribution of their sum?

If $Y_1 \sim \text{Unif}(0,1)$, $Y_2 \sim \text{Unif}(0,1)$ and Y_1, Y_2 independent, then

$$f_{Y_1,Y_2}(y_1,y_2) =$$

To find the density of $U_1 = Y_1 + Y_2$, we will:

1. Identify an auxilary variable $U_2 =$
2. Find the joint pdf of (U_1, U_2).
3. Integrate out u_2 to get marginal pdf of U_1.

$\Rightarrow$ The sum of two indep. Unif(0,1) r.v.'s has a

<u>Picture:</u>