6.5 Method of Moment Generating Functions

- This method uses the fact that the mgf of a r.v. uniquely defines the entire distribution of the r.v.

Theorem: Suppose X, Y are two r.v.'s with mgf's $m_X(t)$ and $m_Y(t)$. If $m_X(t) = m_Y(t)$ for all t, then X and Y have the same probability distribution.

- Suppose U is a function of one or more r.v.'s, i.e., $U \equiv U(Y_1, \ldots, Y_n)$. We can often find the mgf of U, $m_U(t) = E\left(e^{tU}\right)$. If we recognize $m_U(t)$ as being the mgf of some known distribution, then we are assured that U has that distribution.

Example 1: Suppose $Y \sim \text{gamma}(\alpha, \beta)$. What is the distribution of $U = 2Y/\beta$?

- We know the mgf of Y is

$$m_Y(t) =$$

- So the mgf of U is

$m_U(t) =$

- The method of mgf's is especially useful for deriving the distribution of the sum of independent r.v.'s, especially the sum of iid r.v.'s.

<u>Theorem</u>: If $Y_1, Y_2, \ldots, Y_n$ are independent r.v.'s and $U = Y_1 + Y_2 + \cdots + Y_n$, then $M_U(t) = \prod_{i=1}^{n} M_{Y_i}(t)$.

<u>Proof (continuous case)</u>:

Example 2: Suppose $Y_1, \ldots, Y_n$ are independent expon(β) r.v.'s. What is the distribution of $U = Y_1 + Y_2 + \cdots + Y_n$?

Example 3: Suppose $Y_1, \ldots, Y_n$ are independent Poisson r.v.'s with parameters $\lambda_1, \ldots, \lambda_n$. Find the distribution of $U = Y_1 + Y_2 + \cdots + Y_n$.

Example 4: (Important result about linear combination of independent normal r.v.'s)

Suppose $Y_1, \ldots, Y_n$ are independent normal r.v.'s with means $\mu_1, \ldots, \mu_n$ and variances $\sigma_1^2, \ldots, \sigma_n^2$.

What is the distribution of $L = a_1 Y_1 + a_2 Y_2 + \cdots + a_n Y_n$?

Exercise: Use Sec. 6.4, Example 3 and the method of mgf's to show that if $Z_1, Z_2, \ldots, Z_n$ are independent $N(0,1)$ r.v.'s, then $U = Z_1^2 + Z_2^2 + \cdots + Z_n^2$ has a $\chi^2_{(n)}$ distribution.

Note: The method of mgf's works best when the distributions involved have mgf's that are known in closed form.

6.7 Order Statistics

- The order statistics of a random sample $Y_1, \dots, Y_n$ are:

$Y_{(1)}$ = the smallest value in the sample

$Y_{(2)}$ = the second-smallest value in the sample

$\vdots$

$Y_{(n)}$ = the largest value in the sample.

- So $Y_{(1)} \leq Y_{(2)} \leq \cdots \leq Y_{(n)}$. The order statistics themselves are r.v.'s and have probability distributions.

- Why are we interested in the distribution of an order statistic?

Example 1: We are building a seawall to block crashing waves. If $Y_1, \dots, Y_n$ are the heights of n waves, we are interested in the behavior of $Y_{(n)}$, the maximum height.

Example 2: We are building a net to hold particles. If $Y_1, \dots, Y_n$ are the sizes of a sample of particles, we are interested in $Y_{(1)}$, the minimum size.

Note: Some common summary statistics are functions of order statistics:

Sample median =

Sample range =

- The behavior of these statistics in repeated sampling depends on the distributions of order statistics.
- In this section, we will assume the r.v.'s are continuous so that ties in the sample occur with probability 0.

Example 1(a): Suppose $Y_1, \ldots, Y_n$ are iid with pdf $f(y)$ and cdf $F(y)$. What is the pdf for the maximum, $Y_{(n)}$?

- Use the method of cdf's:

Example 1(b): What is the pdf for the minimum, $Y_{(1)}$?

Application 1(a): Suppose wave heights have an exponential distribution with mean height 10 feet. If 200 waves crash during the night, what is the distribution of the highest wave?

Exercise: Find the probability the highest wave is more than 15 feet.

Application 1(b): Suppose light bulbs' life lengths have an exponential distribution with mean 1200 hours. Two bulbs are installed at the same time. What is the expected time until one bulb has burned out?

- Clearly:

- Question: Is the distribution of $Y_{(1)}$ exponential if 3 bulbs are installed here?

Theorem: If $Y_1, \dots, Y_n$ are iid continuous r.v.'s with pdf $f(y)$ and cdf $F(y)$, the pdf of the k-th order statistic, $Y_{(k)}$, is:

Intuition:

- Use multinomial probability argument.
- Factorial terms account for number of different arrangements of data such that $(k-1)$ values $< y$, $(n-k)$ values $> y$.

- Joint distribution of any two order statistics $Y_{(j)}$ and $Y_{(k)}$, $j<k$:

Picture:

Hence:

$$f_{Y_{(j)},Y_{(k)}}(y_j, y_k) =$$

<u>Example 2</u>: 10 numbers are generated at random between 0 and 1. What is the distribution of the 3rd-smallest of these? What is the expected value of the 3rd-smallest?

<u>Note</u>:

<u>Picture</u>:

So $E(Y_{(3)}) =$

<u>Exercise</u>: In the above example, find the joint density of the 5-th and 6-th order statistic. (Using this joint density we could derive the distribution of the <u>sample median</u> here.)