10.10 Power and the Neyman-Pearson Lemma

- Recall that $\beta = P[\text{Type II error}]$

  $=$

- For a particular value of a parameter that is in the alternative region (say θ_a), we can calculate β at θ_a and we can define the power of the test at θ_a as

- In general, the power function of a test is a function of θ and can be defined for any value of θ as

- If we test $H_0: \theta = \theta_0$, then the power function evaluated at θ_0 is

- But for any θ_a in the alternative region, then Power(θ_a) is

which we would like to be ______.

- Since the power at θ_a is $1-\beta$ at θ_a, we can calculate the power of a test similarly as we calculated β back in Section 10.4.
- To calculate the <u>power function</u>, we would have to calculate the power at <u>all</u> values of θ (tedious!) so we can use a computer to help.

<u>Recall Example 1, Sec. 10.4 notes</u>: We tested $H_0: \mu = 500$ vs. $H_a: \mu > 500$ (at $\alpha = .05$) with a z-test. (Assume $\sigma \approx 97.4$).

- We earlier calculated $\beta = .6064$ when $\mu = 515$. So Power(515) =

Picture of Power Function:

- Note that for parameter values in the <u>null</u> region ($\mu \leq 500$), the <u>maximum</u> P[Reject H_0] is achieved at the __________ of the null region (at   ).
- This is generally true for the tests we consider, and it motivates writing $H_0: \mu = 500$ instead of $H_0: \mu \leq 500$.
- Same example, but suppose we test $H_0: \mu = 500$ vs. $H_a: \mu \neq 500$ at $\alpha = .05$.

Picture of Power Function:

- The common tests we study have power functions similar to these examples.
- We see the farther the true parameter value is away from the null value (or region), the ________ power the test has to reject H_0.
- An <u>ideal</u> test would have
  (1) Power $\leq \alpha$ for parameter values in the ________ region
  (2) Power as <u>high</u> as possible for parameter values in the ________ region.
- Such a test is called a <u>most-powerful</u> α-level test.

Simple and Composite Hypotheses

Defn: A statistical hypothesis is called <u>simple</u> if it completely specifies the distribution from which the sample is taken

Defn: A hypothesis that does <u>not</u> completely specify the distribution of the data is called <u>composite</u>.

Example 1: Suppose $Y_1, \dots, Y_n \overset{iid}{\sim} \text{Expon}(\beta)$. Then:

$H: \beta = 2$ is a ________________ hypothesis.

$H: \beta > 2$ is a ________________ hypothesis.

$H: \beta \geq 2$ is a ________________ hypothesis.

Example 2(a): Suppose $Y_1, \dots, Y_n \overset{iid}{\sim} N(\mu, \sigma^2)$ with μ, σ^2 unknown. Then

$H: \mu = 3$ is a ________________ hypothesis.

Example 2(b): Suppose $Y_1, \dots, Y_n \overset{iid}{\sim} N(\mu, \sigma^2)$ with σ^2 known to be 1. Then:

$H: \mu = 3$ is a ________________ hypothesis.

$H: \mu \neq 3$ is a ________________ hypothesis.

The Neyman-Pearson Lemma

- Suppose we are testing a simple null hypothesis versus a simple alternative hypothesis:

$$H_0: \theta = \theta_0 \quad \text{vs.} \quad H_a: \theta = \theta_a$$

where θ_0 and θ_a are two numbers and our data $Y_1, \ldots, Y_n$ are iid from a distribution having parameter θ.

Neyman-Pearson Lemma: For a given α, the test with the highest power at θ_a has a rejection region of the form

where k is a constant chosen to maintain

- The N-P Lemma allows us to find the most-powerful (MP) α-level test for a simple-vs.-simple situation.

Proof (continuous case): Let C be the "rejection region" corresponding to the Neyman-Pearson test; i.e., let C be the set of sample points that lead us to reject H_0. Then the power of the N-P test at θ_a is:

- Let D be the "rejection region" of any other α-level test.

Note:

- The proof in the discrete case is very similar.