10.11 Likelihood Ratio Tests

Note: If $Y_1, \ldots, Y_n \overset{iid}{\sim} N(\mu, \sigma^2)$ with μ and σ^2 both unknown, we cannot use the N-P Lemma for tests about μ because we will not have simple hypotheses with σ^2 unknown.

- An unknown parameter that is not of interest to us (like σ^2 in this example) is called a nuisance parameter.

Defn: The parameter space (denoted by Ω) is the set of values that can be taken by a parameter θ (or vector of parameters $\underset{\sim}{\theta}$).

Example: $N(\mu, \sigma^2)$ distribution:

Gamma (α, β) distribution:

- Let Ω_0 be the (restricted) parameter space implied by the null hypothesis.

<u>Example:</u> $N(\mu, \sigma^2)$, with $H_0: \mu = 0$ vs. $H_a: \mu \neq 0$

- Let $L(\hat{\Omega}_0)$ represent the maximum of the likelihood function based only on parameter values within Ω_0.
- Let $L(\hat{\Omega})$ represent the unrestricted maximum of the likelihood function.
- Recall that the maximum likelihood estimates are the parameter values that "best explain" the data that we observed.
- If $L(\hat{\Omega}_0) = L(\hat{\Omega})$, then the "best explanation for the data is compatible with H_0.
- If $L(\hat{\Omega}_0) < L(\hat{\Omega})$, then the "best explanation for the data lies outside Ω_0 $\Rightarrow$

- A <u>likelihood ratio test</u> is based on comparing $L(\hat{\Omega}_0)$ to $L(\hat{\Omega})$.

Let

be the test statistic.

- The LR test rejects H_0 if

with k chosen so that

<u>Note</u>: The LR method typically works well, but does not always produce UMP tests.

<u>Example 1</u>: If $Y_1, \ldots, Y_n \overset{iid}{\sim} N(\mu, \sigma^2)$ with μ and σ^2 unknown, the LR test of $H_0: \mu = \mu_0$ vs. $H_a: \mu > \mu_0$ is the t-test we studied in Sec. 10.8. See Example 10.24 in the text for details.

Note: All of the tests we obtained in Sections 10.8 and 10.9 (one-sample t-test, two-sample t-test, χ^2-test for a variance, F-test comparing two variances) can be obtained via this LR method.

Example 2: $Y_1, \ldots, Y_n \overset{iid}{\sim} N(\mu, 1)$. Find the LR test of $H_0: \mu = \mu_0$ vs. $H_a: \mu \neq \mu_0$.

- In many cases, it is not easy to find the exact distribution of the LR test statistic.

- We can then use this result to obtain a large-sample LR test:

Theorem: For large n, $-2 \ln \lambda \,\dot\sim\, \chi^2$ with q d.f., where q is the number of extra restrictions on the parameters made by H_0.

__Example 3__: Consider two independent samples $X_1, \ldots, X_m \overset{iid}{\sim} \text{expon}(\theta_1)$ and $Y_1, \ldots, Y_n \overset{iid}{\sim} \text{expon}(\theta_2)$. Find the LR test of $H_0: \theta_1 = \theta_2$ vs. $H_a: \theta_1 \neq \theta_2$.

Application: In this case, if $m=36$, $n=26$, $\sum x_i = 156.8$, $\sum y_i = 133.2$, test $H_0: \theta_1 = \theta_2$ vs. $H_a: \theta_1 \neq \theta_2$ at $\alpha = .05$.