STAT 515 -- Chapter 7: Confidence Intervals

• With a point estimate, we used a single number to estimate a parameter.
• We can also use a <u>set of numbers</u> to serve as "reasonable" estimates for the parameter.

Example: Assume we have a sample of size 100 from a population with $\sigma = 0.1$.

From CLT: $\bar{X}$ has approximately normal sampling distribution with mean μ and std. error $\frac{0.1}{\sqrt{100}} = .01$

Empirical Rule: If we take many samples, calculating $\bar{X}$ each time, then about 95% of the values of $\bar{X}$ will be between: $\mu - .02$ and $\mu + .02$

$\approx$ 95% of the values of $\bar{X}$ will be within .02 units of μ.

Therefore: In $\approx$95% of samples, the unknown population mean μ is within .02 units of $\bar{X}$ (between $\bar{X} - .02$, $\bar{X} + .02$)

This interval $(\bar{X} - .02, \bar{X} + .02)$ **is called an approximate 95% "confidence interval" for μ.**

<u>Confidence Interval</u>: An interval (along with a level of confidence) used to estimate a parameter.

• Values in the interval are considered "reasonable" values for the parameter.

<u>Confidence level</u>: The percentage of all CIs (if we took many samples, each time computing the CI) that contain the true parameter.

<u>Note</u>: The endpoints of the CI are <u>statistics</u>, calculated from sample data. (The <u>endpoints</u> are random, not the parameter!)

In general, if $\bar{X}$ is normally distributed, then in $100(1-\alpha)\%$ of samples, the interval

$1-\alpha = .95$

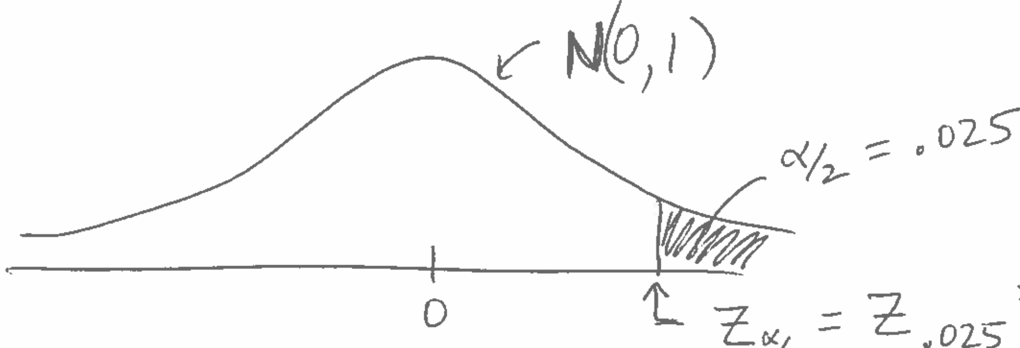
$\alpha = .05$

$\alpha/2 = .025$

$$\left(\bar{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}},\ \bar{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right)$$

will contain μ.

Note: $z_{\alpha/2}$ = the z-value with $\alpha/2$ area to the right:



$z_{\alpha/2} = z_{.025} = 1.96$

(Table III, bottom row)

$100(1-\alpha)\%$ CI for μ: $\bar{X} \pm z_{\alpha/2}(\sigma/\sqrt{n})$

To get $z_{\alpha/2}$ for these CIs, use Table III (t-table), bottom row.

Problem: We typically do not know the parameter σ. We must use its estimate *s* instead.

<u>Formula</u>: CI for μ (when σ is unknown)

Since $\frac{\bar{X}-\mu}{s/\sqrt{n}}$ has a t-distribution with *n* – 1 d.f., our 100(1 – α)% CI for μ is:

$$\bar{X} \pm t_{\alpha/2}\left(\frac{s}{\sqrt{n}}\right)$$

where $t_{\alpha/2}$ = the value in the t-distribution (*n* – 1 d.f.) with α/2 area to the right:

- **This is valid if the data come from a normal distribution.**

⇒ This t CI is approximately valid if n is large and data are not normal.

Example: We want to estimate the mean weight μ of trout in a lake. We catch a sample of 9 trout. Sample mean $\bar{X}$ = 3.5 pounds, *s* = 0.9 pounds. 95% CI for μ?

Could check using a normal Q-Q plot or histogram.

– Let's assume weights are Normally distributed.

$1-\alpha = 0.95 \Rightarrow \alpha = .05, \ \alpha/2 = .025$

For $n-1 = 8$ d.f., $t_{.025} = 2.306$ (Table III)

Left endpoint: $3.5 - (2.306)\left(\frac{0.9}{\sqrt{9}}\right) = 2.81$

Right endpoint: $3.5 + (2.306)\left(\frac{0.9}{\sqrt{9}}\right) = 4.19$

95% CI for μ: (2.81, 4.19)

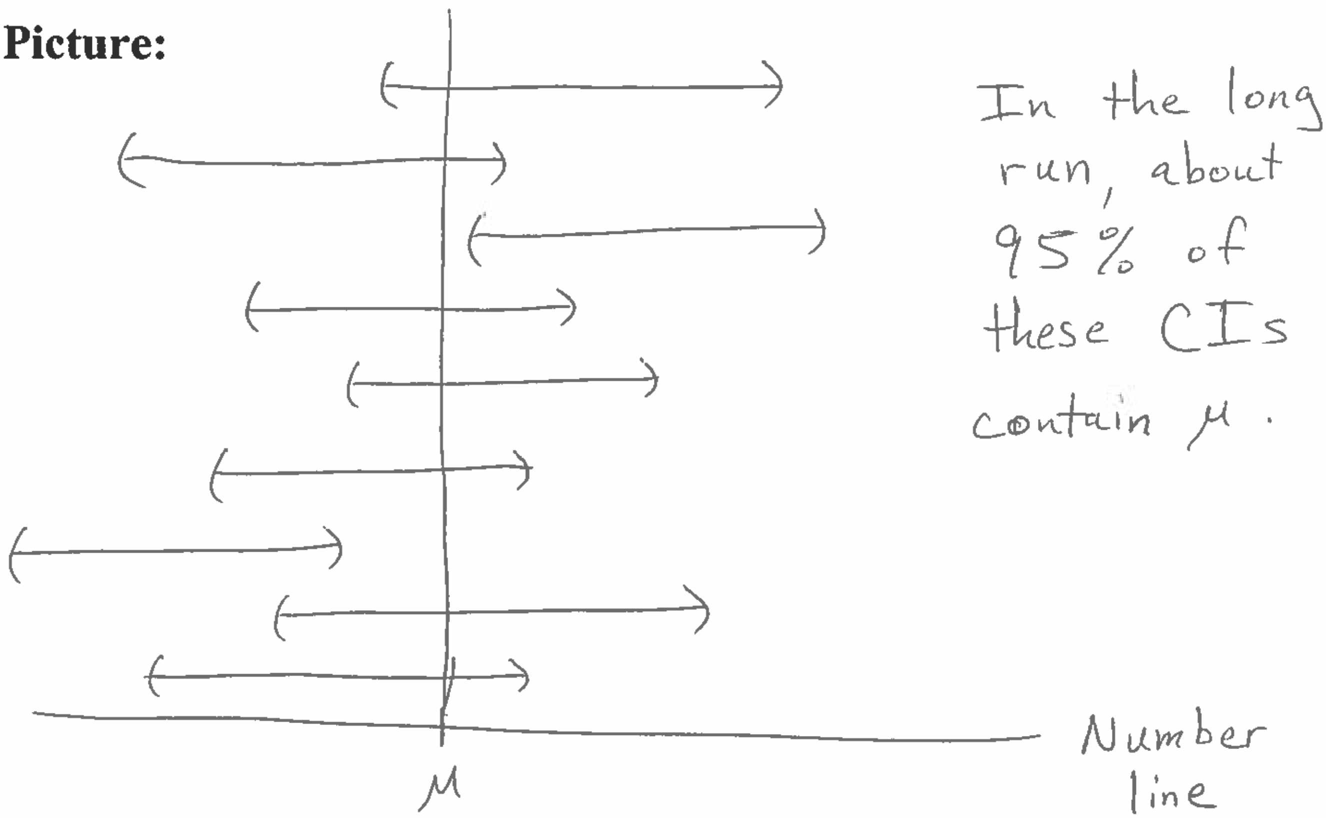
– With 95% confidence, we conclude the true population mean trout weight is between 2.81 and 4.19 pounds.

Question: What does 95% confidence **mean** here, exactly?

• If we took many samples and computed many 95% CIs, then about 95% of them would contain μ.

The fact that (2.81, 4.19) contains μ "with 95% confidence" implies the method used would capture μ 95% of the time, if we did this over many samples.

Picture:



A WRONG statement: "There is .95 probability that μ is between 2.81 and 4.19." Wrong! μ is not random – μ doesn't change from sample to sample. It's either between 2.81 and 4.19 or it's not.

Interpreting a 95% Confidence Interval: TRUE or FALSE?

(1) 95% of all trout have weights between 2.81 and 4.19 pounds. FALSE — CI is about population mean, not about all the trout weights.

(2) 95% of samples have $\bar{X}$ between 2.81 and 4.19.
FALSE — CI is not about sample mean, it's about the population mean.

(3) 95% of samples will produce intervals that contain μ. TRUE

(4) 95% of the time, μ is between 2.81 and 4.19.
FALSE. μ is constant (not random) (it's either b/w 2.81 and 4.19, or it's not)

(5) The probability that μ falls within a 95% CI is 0.95.
TRUE → It is the CI endpoints that are random quantities.

(6) The probability that μ falls between 2.81 and 4.19 is 0.95.
FALSE — The CI is not random once we calculate the specific endpoints.

$1-\alpha = .90, \quad \alpha = .10, \quad \alpha/2 = .05$

$1-\alpha = .99, \quad \alpha = .01, \quad \alpha/2 = .005$

Level of Confidence

Recall example: 95% CI for μ was (2.81, 4.19).

- **For a 90% CI, we use $t_{.05}$ (8 d.f.) = 1.86.**
- **For a 99% CI, we use $t_{.005}$ (8 d.f.) = 3.355.**

} Table III

90% CI: $3.5 \pm (1.86)\left(\frac{0.9}{\sqrt{9}}\right) \Rightarrow (2.94, 4.06)$

95% CI $\longrightarrow (2.81, 4.19)$

99% CI: $3.5 \pm (3.355)\left(\frac{0.9}{\sqrt{9}}\right) \Rightarrow (2.49, 4.51)$

Note tradeoff: If we want a higher confidence level, then the interval gets wider (less precise).

Confidence Interval for a Proportion

- **We want to know how much of a population has a certain characteristic.**
- **The proportion (always between 0 and 1) of individuals with a characteristic is the same as the probability of a random individual having the characteristic.**

Estimating proportion is equivalent to estimating the binomial probability *p*.

Point estimate of *p* is the <u>sample proportion</u>:

$$\hat{p} = \frac{X}{n} = \frac{\#\text{"successes" in sample}}{\#\text{ observations in sample}}$$

Alternative Rule of Thumb to check whether n is large enough:

If $n\hat{p} \geq 15$ and $n\hat{q} \geq 15$, then n is large enough

Note $\hat{p} = \frac{x}{n}$ is a type of sample average (of 0's and 1's), so CLT tells us that when sample size is large, sampling distribution of $\hat{p}$ is approximately normal.

For large *n*:

$$\hat{p} \,\dot{\sim}\, N\left(p, \sqrt{\frac{pq}{n}}\right)$$

where $q = 1 - p$

100(1 – α)% CI for *p* is:

$$\hat{p} \pm z_{\alpha/2} \sqrt{\frac{\hat{p}\hat{q}}{n}}$$

where $\hat{q} = 1 - \hat{p}$

How large does *n* need to be?

When $\hat{p} - 3\sqrt{\frac{\hat{p}\hat{q}}{n}} \geq 0$ <u>and</u> $\hat{p} + 3\sqrt{\frac{\hat{p}\hat{q}}{n}} \leq 1$ then we can use this CI formula

Example 1: A student government candidate wants to know the proportion of students who support her. She takes a random sample of 93 students, and 47 of those support her. Find a 90% CI for the true proportion.

$$\hat{p} = \frac{47}{93} = .505, \quad \hat{q} = 1 - \hat{p} = .495$$

Check: $.505 - 3\sqrt{\frac{(.505)(.495)}{93}} = .505 - .156 = .349 \geq 0$ ✓

$.505 + 3\sqrt{\frac{(.505)(.495)}{93}} = .505 + .156 = .661 \leq 1$ ✓

<u>Can</u> use CI formula!

90% CI $\Rightarrow 1-\alpha = .90 \Rightarrow \alpha = .10 \Rightarrow \alpha/2 = .05$

$z_{.05} = 1.645$ (t-table, bottom row)

Left: $.505 - 1.645\sqrt{\dfrac{(.505)(.495)}{93}} = .420$

Right: $.505 + 1.645\sqrt{\dfrac{(.505)(.495)}{93}} = .590$

90% CI for p: $(.420, .590)$

- With 90% confidence, the true population proportion of students supporting her is between .420 and .590.

Example 2: We wish to estimate the probability that a randomly selected part in a shipment will be defective. Take a random sample of 79 parts, and find 4 defective parts. Find a 95% CI for *p*. $n = 79$

$\hat{p} = \dfrac{4}{79} = .051 \qquad \hat{q} = 1 - \hat{p} = 1 - .051 = .949$

$1-\alpha = .95 \Rightarrow \alpha = .05 \Rightarrow \alpha/2 = .025$

$z_{.025} = 1.96$ (bottom row, t-table)

$0.051 \pm 1.96\sqrt{\dfrac{(.051)(.949)}{79}}$

$\Rightarrow (.0025, .0995)$

With 95% confidence, the probability a randomly selected part is defective is between .0025 and .0995.

- But: $n\hat{p} = 4$ which is not ≥ 15, so we doubt the validity of the CI.

Confidence Interval for the Variance σ^2 (or for s.d. σ)

Recall that if the data are normally distributed,

$\frac{(n-1)s^2}{\sigma^2}$ **has a χ^2 sampling distribution with $(n-1)$ d.f.**

This can be used to develop a $(1-\alpha)100\%$ CI for σ^2:

$$\left(\frac{(n-1)S^2}{\chi^2_{\alpha/2}} , \frac{(n-1)S^2}{\chi^2_{1-\alpha/2}} \right)$$

$(n-1)$ d.f.

normal Q-Q plot

Example: Trout data example (assume data are normal – how to check this?) $s = 0.9$ pounds, so $s^2 =$ $(0.9)^2 = 0.81$
$n = 9$. Find 95% CI for σ^2.

$1-\alpha = .95 \Rightarrow \alpha = .05 \Rightarrow \alpha/2 = .025 \Rightarrow 1 - \alpha/2 = .975$

$n-1 = 8$ d.f. $\rightarrow \chi^2_{.025} = 17.5346, \; \chi^2_{.975} = 2.17973$

(from Table IV)

95% CI for σ^2:

$$\left(\frac{8(0.81)}{17.5346} , \frac{8(0.81)}{2.17973} \right) \Rightarrow \boxed{(.3696, 2.973)}$$

95% CI for σ:

$$\Rightarrow \left(\sqrt{.3696} , \sqrt{2.973} \right)$$

$$\Rightarrow (.608, 1.724)$$

numbers are in terms in "pounds".

Also, a CI for the ratio of two variances, $\frac{\sigma_1^2}{\sigma_2^2}$, can be found by the formula:

$$\left(\frac{s_1^2/s_2^2}{F_{\alpha/2}(n_1-1, n_2-1)}, \frac{s_1^2/s_2^2}{\frac{1}{F_{\alpha/2}(n_2-1, n_1-1)}} \right)$$

Example: If we have a second sample of 13 trout with sample variance $s_2^2 = 0.7$, then a 95% CI for $\frac{\sigma_1^2}{\sigma_2^2}$ is:

$$\left(\frac{0.81/0.7}{3.51}, \frac{0.81/0.7}{1/4.20} \right)$$

$\alpha/2 = .025$

Table VII:

num d.f. = 8

denom d.f. = 12

Table VII:

num. d.f. = 12

denom. d.f. = 8

$(0.33, 4.86)$

Sample Size Determination

book uses the notation "SE"

Note that the **bound** (or **margin of error**) *B* of a CI (about μ or p) equals half its width.

For the CI for the mean (with σ known), this is:

$$B = Z_{\alpha/2}\left(\frac{\sigma}{\sqrt{n}}\right)$$

For the CI for the proportion, this is:

$$B = Z_{\alpha/2}\sqrt{\frac{pq}{n}}$$

Note: When the sample size *n* is bigger, the CI is narrower (more precise).

We often want to determine what sample size we need to achieve a pre-specified margin of error and level of confidence. Solving for *n*:

CI for mean:

$$n = \frac{(Z_{\alpha/2})^2\sigma^2}{B^2}$$

CI for proportion:

$$n = \frac{(Z_{\alpha/2})^2 pq}{B^2}$$

Note: Always round n <u>up</u> to the next largest integer.

These formulas involve σ, p and q, which are usually unknown in practice. We typically guess them based on prior knowledge – often we use $p = 0.5$, $q = 0.5$.

Alternative rule: Guess $\sigma = \frac{\text{Range}}{4}$, where Range = max − min

Example 1: How many patients do we need for a blood pressure study? We want a 90% CI for mean systolic blood pressure reduction, with a margin of error of 5 *mmHg*. We believe that $\sigma = 10$ *mmHg*.

$1-\alpha = .90 \Rightarrow \alpha = .10 \Rightarrow \alpha/2 = .05$

$z_{.05} = 1.645$ (from t-table, bottom row)

$$n = \frac{(1.645)^2 (10)^2}{5^2} = 10.82$$

$\Rightarrow$ Need 11 patients (at least)

Example 2: Pollsters want a 95% CI for the proportion of voters supporting President Obama. Trump **They want a 3% margin of error ($B = .03$). What sample size do they need?** $1-\alpha = .95 \Rightarrow \alpha = .05 \Rightarrow \alpha/2 = .025$

$z_{.025} = 1.96$. Let's use a guess of $p = 0.5$ to play it safe.

$$n = \frac{(1.96)^2 (0.5)(0.5)}{(.03)^2} = 1067.11$$

Need to call 1068 voters (always round up!)