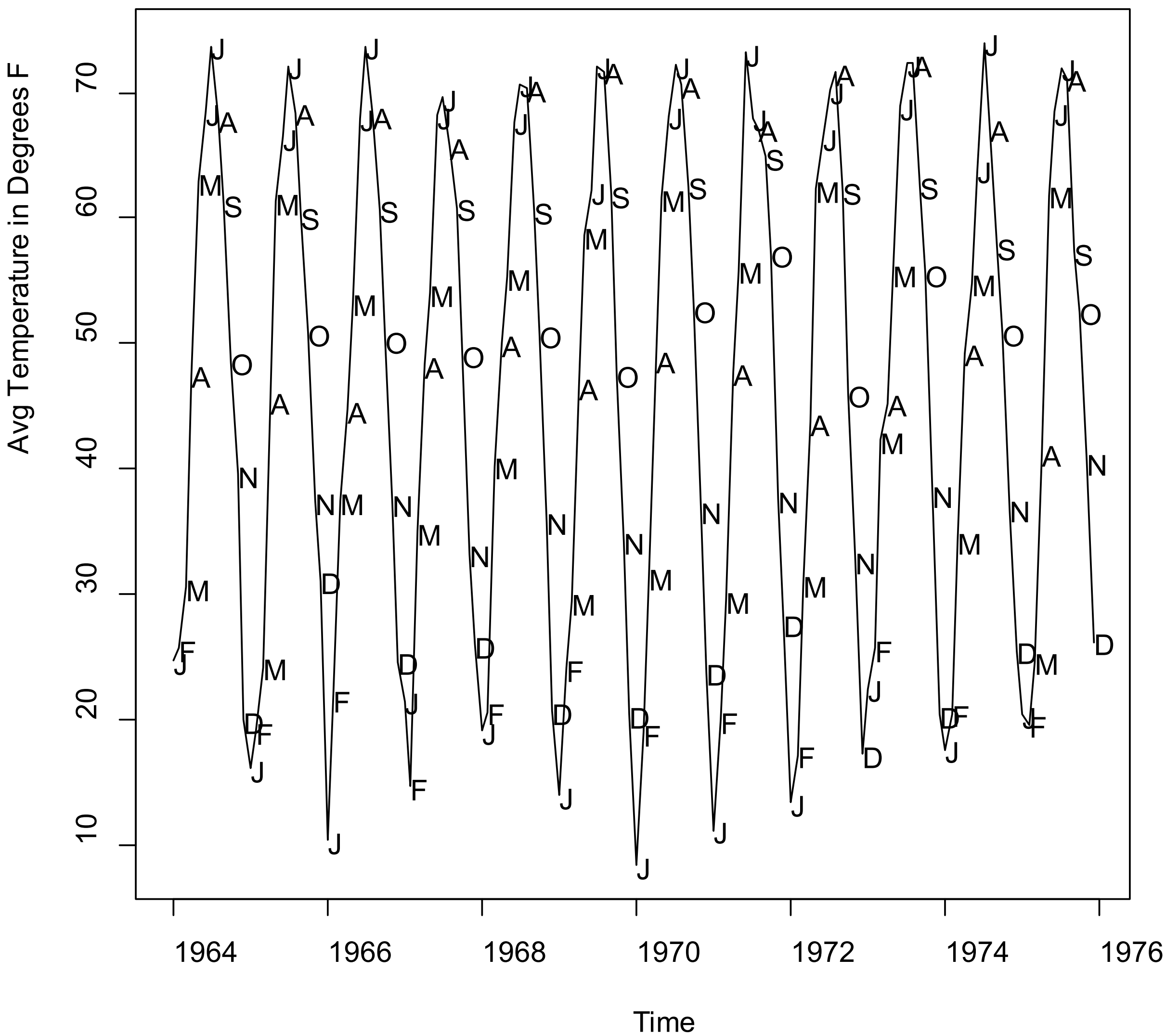
STAT 520 – Homework 1 Example Solutions – Fall 2025

1) Construct a time series plot of the Dubuque temperature data that we studied in class, but include the monthly plotting symbols. Note that the temperature data and the month information are in the `tempdub` object in the `TSA` package. Type `library(TSA); data(tempdub); print(tempdub)` in R to see the data set.

R code:

```
plot(tempdub,type='l',ylab='Avg Temperature in Degrees F')
points(y=tempdub,x=time(tempdub),pch=as.vector(season(tempdub)))
```



2) Consider two random variables, X and Y. Suppose $E(X) = 6$, $var(X) = 9$, $E(Y) = 0$, $var(Y) = 4$, and $corr(X, Y) = 0.25$. Find the following, showing all your steps:
(a) $var(X + Y)$
(b) $cov(X, X + 2Y)$
(c) $cov(4X - 3Y, X + 2Y)$
(d) $corr(X + Y, X - Y)$

Note $corr(X,Y) = 0.25 = cov(X,Y) / [(9)(4)]^{1/2} \Rightarrow cov(X,Y) = (0.25)(6) = 1.5$

(a) $var(X + Y) = var(X) + var(Y) + 2cov(X,Y) = 9 + 4 + 2(1.5) = 16$.
(b) $cov(X, X+2Y) = cov(X,X) + 2cov(X,Y) = 9 + (2)(1.5) = 12$
(c) $cov(4X - 3Y, X + 2Y) = 4cov(X,X) + (4)(2)cov(X,Y) - 3cov(X,Y) - (3)(2)cov(Y,Y)$
$= (4)(9) + (8)(1.5) - (3)(1.5) - (6)(4) = 19.5$
(d) First, $cov(X + Y, X - Y) = cov(X,X) - cov(X,Y) + cov(X,Y) - cov(Y,Y) = 9 - 1.5 + 1.5 - 4$
$= 5$. And note $var(X - Y) = var(X) + var(Y) - 2cov(X,Y) = 9 + 4 - (2)(1.5) = 10$
So $cov(X + Y, X - Y) = cov(X + Y, X - Y) / [var(X + Y) var(X - Y)]^{1/2}$
$= 5 / [(16)(10)]^{1/2} = 0.395$

3) Suppose $\{e_t\}$ is a normal white noise process with mean zero and variance σ_e^2. Let $\{Y_t\}$ be a process defined as: $Y_t = e_t + 0.2e_{t-1}$.
a) Find the autocovariance function *and* autocorrelation function of Y_t.
[Hint: Calculate $cov(Y_t, Y_{t-k})$ case-by-case for several values of k (i.e., for $k = 0$, $k = 1$, $k \geq 2$).] Show all your steps clearly.
b) Is the time series $\{Y_t\}$ stationary? Explain your answer.

a) We calculate $cov(Y_t, Y_{t-k})$ for $k = 0$, $k = 1$, $k \geq 2$:
$cov(Y_t, Y_t) = var(Y_t) = var(e_t + 0.2e_{t-1}) = var(e_t) + 0.2^2 var(e_{t-1}) = \sigma_e^2 + 0.04\sigma_e^2 = 1.04\sigma_e^2$.
$cov(Y_t, Y_{t-1}) = cov(e_t + 0.2e_{t-1}, e_{t-1} + 0.2e_{t-2}) = cov(e_t, e_{t-1}) + 0.2\, cov(e_t, e_{t-2}) +$
$0.2\, cov(e_{t-1}, e_{t-1}) + (0.2)(0.2)\, cov(e_{t-1}, e_{t-2}) = 0 + 0 + 0.2var(e_{t-1}) + 0 = 0.2\sigma_e^2$
For $k \geq 2$, $cov(Y_t, Y_{t-k}) = 0$ since there are no overlapping terms.
So the autocovariance function γ_k is

$$\gamma_k = \begin{cases} 1.04\sigma_e^2 & \text{if } k = 0 \\ 0.2\sigma_e^2 & \text{if } k = 1 \\ 0 & \text{if } k \geq 2 \end{cases}$$

We similarly calculate $corr(Y_t, Y_{t-k})$ for $k = 0$, $k = 1$, $k \geq 2$. Clearly $corr(Y_t, Y_{t-k}) = 1$ for $k = 0$ and $corr(Y_t, Y_{t-k}) = 0$ for $k \geq 2$.
For $k = 1$, $corr(Y_t, Y_{t-1}) = cov(Y_t, Y_{t-1}) / [var(Y_t) var(Y_{t-1})]^{1/2} = 0.2\sigma_e^2 / [1.04\sigma_e^2 1.04\sigma_e^2]^{1/2}$
$= 0.2 / [1.04] = 0.1849$
since $var(Y_t) = 1.04\sigma_e^2$ as shown above, and $var(Y_{t-1}) = 1.04\sigma_e^2$ by a similar argument.
So the autocorrelation function ρ_k is

$$\rho_k = \begin{cases} 1 & \text{if } k = 0 \\ 0.1849 & \text{if } k = 1 \\ 0 & \text{if } k \geq 2 \end{cases}$$

(b) Yes, $\{Y_t\}$ is stationary. The mean function $E(Y_t) = 0$ and does not depend on t. The variance function $var(Y_t) = 1.04\sigma_e^2$, which does not depend on t. And the autocovariance function $\gamma_k = cov(Y_t, Y_{t-k})$ depends only on the lag k and not on t. So $\{Y_t\}$ is weakly stationary, and since $\{Y_t\}$ is also normal, it is stationary.

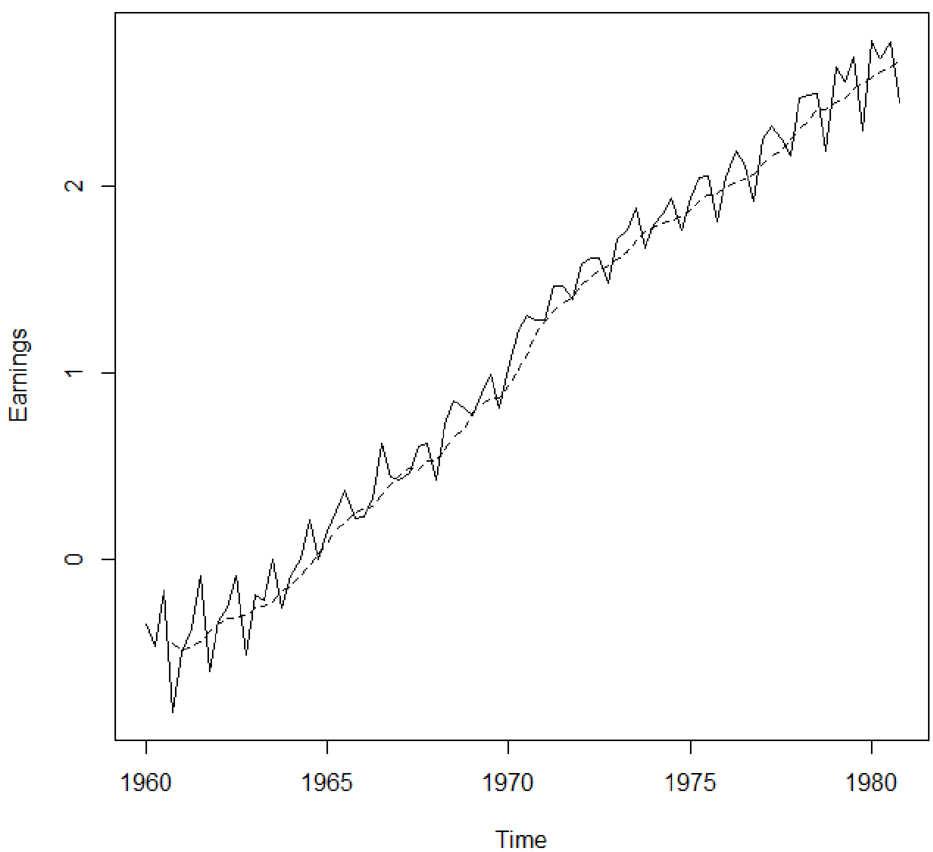
4) Apply a moving average filter to Y_t, where Y_t is the natural logarithm of the Johnson and Johnson earnings data (the original data are given in the `jj` object in the `astsa` package). Specifically, let $V_t = (Y_t + Y_{t-1} + Y_{t-2} + Y_{t-3}) / 4$. The R code

```
v = filter(y, rep(1/4, 4), sides = 1)
```

may be helpful in implementing this. Type `help(filter)` in R for more details about this R function. Plot Y_t as a line and overlay (superimpose) V_t as a dashed line, and provide this plot. Discuss whether the moving average filter captures the overall trend in the time series.

R code:

```
library(astsa)
y = log(jj)
v = filter(y, rep(1/4,4), sides=1)
plot(y, type="l", ylab="Earnings")
lines(v, lty=2)
```



We see that the moving average filter captures the basic increasing trend of the plot quite well. The mean of the logged earnings increases almost linearly, but less steeply in the early 1960s and more steeply in the years around 1970.

5) [Required for graduate students, extra credit for undergraduate students] Suppose $\{e_t\}$ is a normal white noise process with mean zero and variance σ_e^2. Let $\{Y_t\}$ be a process defined as:
$Y_t = e_t e_{t-1}$. Showing all your steps, find the mean function and the autocovariance function of Y_t. [Hint: Use facts about the expected value of the product of independent random variables.] Is the time series $\{Y_t\}$ stationary? Explain your answer.

Note that the $\{e_t\}$ values are independent.
$\mu_t = E[e_t e_{t-1}] = E[e_t]E[e_{t-1}] = (0)(0) = 0$ for all t.

$\text{cov}(Y_t, Y_t) = \text{var}(Y_t) = E[(Y_t - \mu_t)^2] = E[Y_t^2] = E[e_t^2 e_{t-1}^2] = E[e_t^2]E[e_{t-1}^2] = \text{var}[e_t]\text{var}[e_{t-1}] =$
$(\sigma_e^2)(\sigma_e^2) = \sigma_e^4$.

$\text{cov}(Y_t, Y_{t-1}) = \text{cov}[e_t e_{t-1}, e_{t-1} e_{t-2}] = E[e_t e_{t-1} e_{t-1} e_{t-2}] - E[e_t e_{t-1}]E[e_{t-1} e_{t-2}]$
$= E[e_t e_{t-1}^2 e_{t-2}] - E[e_t]E[e_{t-1}]E[e_{t-1}]E[e_{t-2}] = E[e_t]E[e_{t-1}^2]E[e_{t-2}] - [0][0][0][0]$
$= [0]\ E[e_{t-1}^2]\ [0] - 0 = 0$

For $k \geq 2$, $\text{cov}(Y_t, Y_{t-k}) = \text{cov}[e_t e_{t-1}, e_{t-k} e_{t-k-1}] = 0$ using the same approach as above and noting there will be no subscripts in common when $k \geq 2$.

So

$$\gamma_k = \begin{cases} \sigma_e^4 & \text{if } k = 0 \\ 0 & \text{if } k = 1 \\ 0 & \text{if } k \geq 2 \end{cases}$$

The process is weakly stationary since the mean function does not depend on time, and for any k, the autocovariance function γ_k does not depend on time. And furthermore, since the process is normal, it is stationary.